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IT FROM QUBIT OR ALL FROM HALL?

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Generalized 1 + 0-dimensional Liouvillean dynamics describing deformations of the Sachdev–Ye–Kitaev (SYK) model, as well as the various 1 + 1-dimensional dilaton and Horava–Lifshitz gravity theories, can all be mapped onto the single-particle quantum mechanics of a non-relativistic charge propagating in a (generally, curved) 2d space and subject to a (generally, non-uniform) magnetic field. The latter description sets a stage for the phenomenon of quantum Hall effect (QHE), thereby elucidating the intrinsically topological nature of the pertinent gravity theories and demystifying their (pseudo)holographic connection to a broad class of the SYK-like models.

Keywords: qubit, SYK model, Liouville theory, Horava–Lifshitz gravity

1. Holographic mirages

In light of the slower-than-desired progress in understanding of the great many quantum many-body systems there has long been a dire need for finding a universal geometric (or, possibly, hydrodynamic) description of interacting quantum (fermionic) matter in terms of some semi-classical collective field variables.

Historically, that idea was first implemented in the framework of classical kinetic theory formulated in terms of the Wigner distribution function and its moments, the description of which could then be further advanced to the (formally exact) phase space path integral over the corresponding field variable. Conceptually, such a construction can be classified as Kostant–Kirillov co-cycle quantization on the orbits of a given system's dynamical symmetry group.

However, the intrinsic complexity of working with such exact, yet intractable formalism brought out a variety of approximate techniques, and the best known one is 'ad hoc' bosonization by which the quantum dynamics of interacting matter would be accounted for in terms of the shape fluctuations of the corresponding Fermi surface [1–19].

Albeit being quite different in its appearance, the more recent conjecture of holographic duality has been pursuing a similar goal. In this novel proposal, the equivalent bosonic variables would be assumed to organize into multiplets reminiscent of the metric, vector, and scalar fields in one higher dimension and governed by some local Einstein–Maxwell-scalar type of action (see Refs. [20–26] and references therein).

Although vigorous attempts to put the general holographic conjecture on a solid ground have been continuing for over two decades, a satisfactory proof still remains elusive. This fact notwithstanding and putting the general burden of proof aside, much of the massive effort exercised under the auspices of the so-called AdS/CMT (Anti-de-Sitter/condensed matter theory) branch of applied holography has been devoted to the heuristic 'bottom-up' approach [20–26].

This approach unequivocally postulates the validity of the holographic conjecture in its broadest interpretation while making the specific choice of a dual gravity theory and its bulk metric largely on the basis of technical convenience. However, judging by the mere number of publications [27], this straightforward approach has been undergoing a notable demise, as of lately. Thus, the time may has come to pursue the holographic endeavour in a better grounded and more systematic approach.

To that end, the attempts to construct a general holographic picture from the first principles of quantum information (e.g. tensor networks) implementing the popular 'IT from QUBIT' paradigm [20–26], so far, have not proceeded beyond the exploratory level. In most cases where the bulk metric was definitively ascertained, it was found to be of the basic AdS type, thus casting doubts on the possibility of constructing anything as exotic as, e.g. the helical 'Bianchi *VII*' geometry that has been repeatedly invoked in the AdS/CMT scenarios of the so-called 'strange metallic' normal state of cuprates [28, 29].

Also, the previous attempts to derive holography directly from the scale-dependent renormalization group (RG) flow, thus implementing the holographic 'RG=GR' principle [20–26], remain largely inconclusive and, thus far, produced either the plain AdS or, else, unrecognizable bulk geometries. Likewise, the attempts [30, 31] to establish a holographic correspondence between the bulk AdS gravity (in the Lorentz signature) and an ordinary superconductor (even a weakly coupled BCS one) hinge on a formal similarity between the 2d d'Alembertian operator acting in the so-called kinematic space and the mixed second derivative of a bilocal function (see, e.g. Eq. (19) below), thus falling short of providing any actual 'proof' of holography.

As compared to all the questionable (and, for its most part, easily refutable [32–37]) evidence that was purportedly consistent with AdS/CMT, the recent studies of the holography-like correspondence [38–55] between the ensemble-averaged quantum mechanical SYK model [56–66] in 1 + 0 dimensions and Jackiw–Teitelboim (JT) gravity [67–88] in 1 + 1 dimensions may seem to have finally delivered a strong argument supporting the holographic conjecture (albeit in the form that is quite different from all the earlier 'ad hoc' AdS/CMT constructions [20–26]).

At the very minimum, the following discussion aims at extending the list of the holographically dual 1 + 0- and 1 + 1-dimensional problems beyond the extensively studied case of SYK-JT. It will be argued that this specific example represents the more general equivalence between a whole class of the deformed SYK models and a certain family of generalized 2d gravities.

Even more importantly, taken at its face value the SYK-JT duality raises an important question as to whether or not any (or all) instances of actually proven – as opposed to the merely assumed – cases of holo-graphic correspondence would be limited to those situations where the bulk theory appears to be of a (possibly, implicit) topological nature?

In the specific case of SYK-JT, the bulk system does happen to be intrinsically topological, akin to quantum Hall effect (QHE). Therefore, should the answer to the above question happen to be affirmative, it would naturally explain the otherwise rather baffling duality between some systems of (ostensibly) different dimensionalities, as per the central holographic conjecture. Also, it would prompt one to look for the hidden 'Hallness' in those situations where some holographic features may have been observed.

2. From SYK to Liouville via Schwarzian

Extensions of the original SYK model are described by a generic Hamiltonian

$$\hat{H} = \sum_q \sum_{i_1 \ldots i_q} J_{i_1 \ldots i_q} \hat{\chi}_{i_1} \ldots \hat{\chi}_{i_q}, \tag{1}$$

which combines the products of some even number q of the N-coloured Majorana or Dirac fermion operators $\hat{\chi}_i(\tau)$, where $i = 1,\ldots, N$ [56–66]. In turn, the independent Gaussian-distributed classical random amplitudes $J_{i1\ldots iq}$ of the all-to-all q-body entanglement are characterized by the variances

$$\overline{J_{i_1 \ldots i_q} J_{j_1 \ldots j_q}} = J_q^2 \prod_{k=1}^{q} \delta_{i_k, j_k}. \tag{2}$$

The analysis of the model (1) typically starts by integrating the fermions out, thereby arriving at the action in terms of the bilocal field $G(\tau_1, \tau_2)$, which represents the fermion propagator and the corresponding self-energy $\Sigma(\tau_1, \tau_2)$ [56–66]

$$S[G,\Sigma] = \frac{N}{2} \int \mathrm{d}\tau_1 \int \mathrm{d}\tau_2 [\ln(\partial \tau_1 \delta(\tau_1 - \tau_2) - \Sigma(\tau_1, \tau_2)) + \Sigma(\tau_1, \tau_2) G(\tau_1, \tau_2)] - F[G(\tau_1, \tau_2)], \tag{3}$$

where the functional $F[G]$ results from the Gaussian averaging. Moreover, Eq. (2) can be further promoted to a retarded and/or non-uniform disorder correlation function, thus introducing a notion of spatial dimensions and space/time-dependent (retarded and/or nonlocal) entanglement-like couplings [89, 90].

Solving for the self-energy $\Sigma = \delta F/\delta G$, the Schwinger–Dyson equation derived from Eq. (3) can be cast in the form

$$\int d\tau \left(\partial_\tau \delta(\tau_1 - \tau) + \frac{\delta F}{\delta G(\tau_1, \tau)} \right) G(\tau_1, \tau_2) = = \delta(\tau_1 - \tau_2). \tag{4}$$

In the original SYK$_q$ model with $F[G] = J^2 G^q$, Eq. (4) remains invariant under the infinite group $Diff(S^1)$ of reparametrizations (diffeomorphisms) of the thermal circle $\tau \to f(\tau)$ with the periodicity condition $f(\tau + \beta) = f(\tau) + \beta$, as long as the derivative term is neglected and provided that G and Σ transform as

$$G_f(\tau_1, \tau_2) = [\partial_{\tau_1} f(\tau_1)\, \partial_{\tau_2} f(\tau_2)]^{\Delta} G(f(\tau_1), f(\tau_2)),$$

$$\Sigma_f(\tau_1, \tau_2) = [\partial_{\tau_1} f(\tau_1)\, \partial_{\tau_2} f(\tau_2)]^{1-\Delta} \Sigma(f(\tau_1), f(\tau_2)). \tag{5}$$

The above properties of Eq. (4) single out a translationally-invariant 'conformal' mean-field solution (hereafter, $\tau = \tau_1 - \tau_2$ and β are the inverse temperature) [38–66]

$$G_0(\tau_1, \tau_2) = \left(\frac{\pi}{\beta \sin(\pi\tau/\beta)} \right)^{2\Delta}. \tag{6}$$

In the zero-temperature limit and for $J\tau \gg 1$, it demonstrates a pure power-law behaviour $G_0(\tau_1, \tau_2) \sim 1/(J\tau)^{2\Delta}$ with the fermion dimension $\Delta = 1/q$.

This solution spontaneously breaks the full reparametrization symmetry down to its three-dimensional subgroup $SL(2, R)$ implemented through the Möbius transformations $\tau \to (a\tau + b)/(c\tau + d)$, where $ad - bc = 1$, under which the solution (6) and the action (3) remain invariant.

The reparametrization transformations outside the $SL(2, R)$ subgroup modify the functional form of G_0, thus exploring the entire coset $Diff(S^1)/SL(2, R)$ and providing it with the structure of a coadjoint Virasoro orbit.

The deviations from Eq. (6) are controlled by the short-time expansion

$$\delta G_f(\tau_1, \tau_2) = = \frac{\Delta}{6} \tau^2 Sch\{f(T), T\} G_0^2(\tau_1, \tau_2) + \dots . \tag{7}$$

Hereafter $T = (\tau_1 + \tau_2)/2$ and Sch denote the Schwarzian derivative, $Sch\{f, x\} = f'''/f' - \frac{3}{2}(f''/f')^2$ (here $f' = df/dx$), which obeys the differential 'composition rule' $Sch\{F(f), x\} = Sch\{F(f), f\} f'^2 + Sch\{f, x\}$. The dynamics of the variable $f(\tau)$ is then governed by the non-reparametrization invariant, yet manifestly geometrical and $SL(2, R)$-invariant, action

$$S_0[f] = -\frac{N}{Jq^2} \int d\tau\, Sch\{\tan \frac{\pi f}{\beta}, \tau\} \tag{8}$$

that stems from the trace of the (infrared-irrelevant in the RG sense) time derivative $\partial_\tau G$ in the gradient expansion of the first term in Eq. (3).

The mean-field ('large-N') SYK solution (6) is only applicable for $1/J \ll \tau, \beta \ll N/J$, and under such conditions the fluctuations δG about the saddle point G_0 remain small. By contrast, in the 'Schwarzian' (long-time, low-temperature, $N/J \lesssim \tau, \beta$) limit these fluctuations can grow strong, thereby significantly altering the mean-field behaviour [38–55].

Namely, upon the Langer transformation $\partial_\tau f = e^{\phi}$ the Schwarzian action (8) reduces to the ostensibly free expression in terms of the unbounded variable $\phi(\tau)$, $S_0[\phi] \sim d\tau(\partial_\tau \phi)^2$. However, the true Liouvillean action remains strongly non-Gaussian, as follows from the analysis of the products of propagators:

$$\langle G_f(\tau_1, \tau_2) \dots G_f(\tau_{2p-1}, \tau_{2p}) \rangle = = \int D\phi\, e^{-S_0[\phi]} \prod_{i=1}^{p} \frac{e^{\Delta(\phi(\tau_{2i-1}) + \phi(\tau_{2i}))}}{(\int_{\tau_{2i-1}}^{\tau_{2i}} d\tau e^{\phi})^{2\Delta}}. \tag{9}$$

Computing such amplitudes requires the denominator to be promoted to the exponent in the form of the $2p$ consecutive quenches under the action of the local 'vertex' operators $e^{\Delta\phi(\tau)}$. The resulting effective action,

$$S[\phi] = \frac{N}{Jq^2} \int d\tau \left(\frac{1}{2} (\partial_\tau \phi)^2 + J^2 e^{2\phi} \right), \tag{10}$$

then acquires the exponential 1d Liouville potential $V_{2,2}(\phi) = J^2 e^{2\phi}$ (hereafter, $V_{a,b}(\phi)$ denotes a potential which behaves as $e^{(a/b)\phi}$ in the limits $\phi \to \pm\infty$, respectively).

The action $S[\phi]$ can be quantized by switching to the Hamiltonian picture and further substituting the momentum $\pi = \delta S/\delta\partial_\tau\phi$ with $-\mathrm{i}\partial_\phi$.

Converting the action (10) to the 1d Hamiltonian and further quantizing it by virtue of the standard substitution $\partial_\tau\phi \to -\mathrm{i}\partial_\phi$ one arrives at the (static) eigenvalue equation

$$-\partial_\phi^2\psi + V_{2,2}(\phi)\psi = E\psi. \tag{11}$$

The spectrum of Eq. (11) is continuous, $E_k = k^2$, and consists of the eigenstates $\psi_k(z) \sim K_{2\mathrm{i}k}(2\sqrt{z})$, where $z = J\beta e^\phi$. These exact wave functions can be used to compute the various matrix elements $\langle 0|e^{\Delta\phi}|k\rangle$ explicitly. This calculation reveals a universal behaviour of the averaged products of an arbitrary number of propagators in the long-time/low-temperature regime ($N/J \lesssim \tau, \beta$) where one finds $\langle G_f^p(\tau, 0)\rangle \sim 1/(J\tau)^{3/2}$ for any $p \geq 1$ and $q \geq 2$ [51, 52]. This behaviour is markedly different from the (non-universal) mean-field one at short times/high temperatures ($1/J \ll \tau, \beta \ll N/J$), $\langle G_f^p(\tau)\rangle \sim \langle G_0^p(\tau)\rangle \sim 1/J\tau^{2p/q}$.

The intrinsically non-Gaussian nature of the action (10) is manifest. Otherwise, the latter amplitude would have been governed by the logarithmic correlator $\langle\phi(\tau)\phi(0)\rangle_G \sim J\tau$, thus demonstrating an exponential, rather than algebraic, decay, $\langle G_f^p(\tau)\rangle_G \sim G_0^p(\tau)\,\exp(-\frac{1}{2}p^2\Delta^2\langle\phi(\tau)\phi(0)\rangle_G)$, which is also non-universal as a function of p and q.

Notably, the 1d action (10) is for a single variable representing the fluctuations of a single soft (energy) mode. It can be readily extended to include other degrees of freedom – as, e.g. in the case of the complex-valued ('Dirac', as opposed to 'Majorana') variant of the SYK model, the additional scalar field corresponding to the charge fluctuations [56–66].

3. SYK deformations

A deformation of the 'potential' part of the Liouvillean action (10) can generally be represented in terms of a two-time integral with the kernel

$$F[G] = \sum_n c_n \int \mathrm{d}\tau_1 \int \mathrm{d}\tau_2 G^n(\tau_1, \tau_2), \tag{12}$$

where $c_n \sim NJ_n^{\,2}/q^2$, which results from the ensemble-averaged partition function and generalizes the original SYK model described by the single $n = q$ term. Additional powers of G could also emerge if random amplitudes of the n- and m-particle terms developed some (physically quite plausible) cross-correlations, resulting in $\overline{J_n J_m} \neq 0$.

Beyond the Liouville point in the multi-dimensional Hamiltonian parameter space, the previous analyses of the action given by Eqs. (3) and (12) have been largely limited to the SYK_q-$\mathrm{SYK}_{q/2}$ model with only two non-zero coefficients, $c_q = J^2N/2q^2$ and $c_{q/2} = 2\Gamma^2N/q^2$. For $q = 4$, it has been rather extensively discussed in the context of random tunnelling between two SYK quantum dots [91–104], the amplitude Γ being a variance of the tunnelling amplitude. This action also finds its applications in theoretical cosmology ('traversable wormhole') and discussions of the 1 + 1-dimensional analog of the Hawking–Page curve [105, 106].

Moreover, in most of the previous analyses the terms with $n = q/2$ and $n = q$ would be treated, respectively, as small perturbations of one another. Specifically, at relatively short times $1/J \ll \tau \ll J/\Gamma^2$, the value of the fermion dimension $\Delta = 1/q$ would be determined by the $n = q$ term, while for $\tau \gg J/\Gamma^2$ the $n = q/2$ term takes over, thus causing a faster decay governed by $\Delta = 2/q$.

Such analysis can be potentially misleading, as it focuses on the soft ('angular' or 'along-the-valley') fluctuations about a chosen mean-field solution, while under a perturbation the mean-field solution itself might undergo a significant change which would then require a tedious account of the hard ('radial' or 'out-of-the-valley') fluctuations. It can be avoided, though, by using the proper solution of the mean-field equation derived for the entire functional (12).

Of a particular interest are the crossovers between different conformal fixed points where all pertinent coupling constants are of the same order. Such 'SYK transits' are not directly amenable to perturbation theory in the vicinity of the fixed points in question but can still be explored in the large-q limit (see the next Section). To that end, one can utilize the already available and seek out new – non-perturbative (in general, non-conformal) mean-field solutions which interpolate between different conformal regimes [107, 108].

By analogy with the pure Liouvillean action (10), the canonical quantization procedure applied to the action $S_0 + \Delta S$ given by the sum of Eqs. (8) and (12) substitutes its non-Gaussian part with the ordinary single-time integral $\Delta S(\phi) = \int d\tau V(\phi)$, where

$$V(\phi) = \sum_n c_n e^{2n\phi/q}. \quad (13)$$

For one, the aforementioned two-term action with the non-zero coefficients c_q and $c_{q/2}$ features the Morse potential [38–55, 107, 108]

$$V_{2,1}(\phi) = A_q e^{2\phi} + c_{q/2} e^{\phi}. \quad (14)$$

For both c_q and $c_{q/2}$ positive the potential (14) features a continuous positive definite spectrum, $E_\nu = \nu^2 + \lambda^2 + 1/4$, with the eigenstates

$$\psi_\nu(\phi) \sim \sqrt{\nu \sinh(2\pi\nu)}\Gamma(1/2 - \lambda + i\nu) \times W_{\lambda, i\nu}(2\lambda e^{\phi}), \quad (15)$$

where $\lambda = c_{q/2}/c_q$ and $W_{\lambda,\, i\nu}$ is the Whittaker function. For $c_{q/2} = 0$ Eq. (15) reduces to the eigenstates of the Liovillean potential given by the modified Bessel functions.

By contrast, for $c_{q/2}$ negative the potential develops a minimum and the spectrum includes $\mathcal{N} = [\lambda - 1/2]$ bound states at the negative energies $E_n = \lambda^2 + 1/4 - (n - \lambda + 1/2)^2$, where $n = 0,\dots, \mathcal{N}$. The corresponding eigenstates are given by the associated Laguerre polynomials

$$\psi_n(\phi) \sim e^{(\lambda-n-1/2)\phi - e^{\phi}/2} L_n^{2\lambda-2n-1}(2\lambda e^{\phi}). \quad (16)$$

At low temperatures ($\Gamma^2\beta/J \gg 1$), the number $\mathcal{N}$ of bound states increases and they become nearly equidistant, as in the harmonic oscillator potential.

Notably, for $J = 2\Gamma$ the aforementioned monotonic and non-monotonic Morse potentials organize into a doublet of super-partners $V_\pm(\phi) = W^2(\phi) \pm \partial_\phi W(\phi)$ with $W(\phi) \sim e^{\phi}$, which conspire into one supersymmetric potential. The ground state of the binding potential V_- then takes on the form $\psi_0(\phi) \sim \exp(-\int W d\phi)$.

Conceivably, the effective action $S(\phi)$ may develop other interesting regimes at the points of still higher symmetry. Albeit being special, the integrable potentials may also provide insight into the general behaviours. A similar situation has long been known in the physics of integrable 1d spin chains of arbitrary on-site spin.

In particular, below we demonstrate the emergence of the Toda-like action (in cosmology, a.k.a. 'oscillatory tracker model') described by the classically solvable two-term potential

$$V_{2,-2}(\phi) = c_q e^{2\phi} + c_{-q} e^{-2\phi}, \quad (17)$$

which has only discrete levels. Its linearly independent solutions are given by the approximate formulas $\psi_\pm(\phi) \sim \exp(\pm e^{\phi} - \phi/2)$. Notably, both Eqs. (14) and (17) belong to the still broader family of 'quasi-solvable' potentials $V(\phi) = c_q e^{2\phi} + c_{q/2} e^{\phi} + c_{-q/2} e^{-\phi} + c_{-q} e^{-2\phi}$.

In the problem of tunnelling between two SYK quantum dots, going into the strong-coupling regime and taking into account multiple tunnelling processes can be achieved by replacing G computed to zeroth order in tunnelling with the all-order expression $G/(1 + i\sigma G)$, where σ is the tunnelling conductance [91–104].

The corresponding potential $V(\phi)$ can then consist of an infinite number of terms. In that regard, especially interesting is the 'hypersymmetric' Hulten potential $V_{0,1}(\phi) \sim e^{\phi}/(1 - e^{\phi})$ with all the coefficients $c_{nq/2} = c$ for all $n \geq 1$. It develops the $\sim 1/\phi$ behaviour at small ϕ, reminiscent of the Coulomb potential. Unlike the latter, though, it features only a finite number ($[\lambda]$) of bound states at $E_n = -[(\lambda^2 - n^2)/2\lambda n]^2$.

Another interesting ('variable scaling') model was proposed in Refs. [109–112]. It includes an infinite number of terms with the coefficients $c_{nq/2} \sim n^\nu$. Performing an approximate summation over n one obtains a power-law potential $V_{\infty,0}(\phi) \sim \sum_n n^{\nu-1} e^{n\phi} \sim \frac{1}{(-\phi)^\nu}$ generalizing the Coulomb one.

4. Large *q* limit

An alternate approach to the generalized SYK-like models and a further justification of substituting Eq. (13) for Eq. (12) exploits the large-q approximation to the propagator [38–55]

$$G(\tau_1, \tau_2) = \frac{1}{2} \operatorname{sgn} \tau \left(1 + \frac{2}{q} g(\tau_1, \tau_2) + \dots \right). \quad (18)$$

The higher order terms $O(1/q^2)$ can also be evaluated, albeit at increasingly prohibitive costs. The path integral over the field g is governed by the action

$$S(g) = \frac{N}{q^2} \int d\tau_1 \int d\tau_2 \left(\frac{1}{2} \partial_{\tau_1} g \partial_{\tau_2} g + V(g) \right), \quad (19)$$

where the potential is given by Eq. (13) as a function of the bilocal field $g(\tau_1, \tau_2)$.

A complete theory (19) is genuinely two-dimensional, the relative τ and 'center-of-mass' $T = (\tau_1 + \tau_2)/2$ time variables playing the roles of the effective 'radial' and 'temporal' coordinates in the 2d 'kinematic space', respectively [113–117]. So it is only by focusing on the former dependence and neglecting the latter can one reduce the low-energy sector of Eq. (19) to the 1d action akin to that given by Eqs. (8) and (13).

This way, one arrives at the equation of motion

$$\partial_\tau^2 g(\tau) = -\partial_g V(g(\tau)), \qquad (20)$$

the solutions of which correspond to the mean-field configurations, thus yielding the mean-field propagator $G_0(\tau) = \exp(2g(\tau)/q)$.

A solution to Eq. (20) provides one with the means to probe the system's thermodynamics. To that end, by solving Eq. (20)

$$\tau = \int_{g_0}^{0} \frac{\mathrm{d}g}{\sqrt{V(g_0) - V(g)}} \qquad (21)$$

and putting $\tau = \beta/2$ one computes the mean-field energy [109]

$$E = \frac{N}{4q^2}\left[\beta V(g_0) - 2^{3/2}\int_{g_0}^{0} \mathrm{d}g\sqrt{V_0 - V(g)}\right], \qquad (22)$$

where $g_0 < 0$ is the turning point of the potential $V(g)$.

In the case of the Morse potential (14) with g substituted for ϕ, the explicit saddle point solution of Eq. (20) reads [109–112]

$$g_0(\tau) = \ln \frac{2A\sin^2\theta}{\cos(2\omega\tau/\beta - \omega) + \cos\theta}, \qquad (23)$$

where $A = \sqrt{(\omega/\beta J)^2 + (\Gamma/J)^4}$, $\theta = \tan^{-1}(\omega J/\beta\Gamma^2)$, and the ω obeys the equation $2\omega^2 = (\beta\Gamma)^2 + A(\beta J)^2 \cos\omega$. It takes the values $\omega = \pi/2 - O(1/\beta J)$ and $\omega = \pi/2 - O(\Gamma^2\beta/J)$ for $\beta \ll 1/\Gamma$ and $\beta \gg 1/\Gamma$, respectively.

In the zero-temperature limit, Eq. (18) yields

$$G_0(\tau) = \frac{1}{2}\frac{\operatorname{sgn}\tau}{(1 + \sqrt{J^2 + 4\Gamma^2}\tau + \Gamma^2\tau^2)^{2/q}}. \qquad (24)$$

As compared to the approximate conformal propagator characterizing the original SYK model, this expression is UV-finite and naturally regularized at $\tau \lesssim \min\,[1/J, 1/\Gamma]$. Also, in contrast with the perturbative results of Refs. [91–102], the saddle-point solution (24) is applicable at all Γ/J, large and small.

Gaussian fluctuations $\delta g(\tau)$ about the saddle-point solution of Eq. (20) are governed by the action

$$\delta S = \frac{N}{2q^2}\int \mathrm{d}\tau[(\partial_\tau \delta g)^2 + W(g_0(\tau))\delta g^2] \qquad (25)$$

featuring the potential $W(g(\tau)) = \partial_g^2 V(g) = \sum_n c_n n^2 e^{ng}$, which is functionally similar to $V(g)$ given by Eq. (13) and has to be evaluated at the solution $g_0(\tau)$ of Eq. (20).

In contrast to the Schwarzian action (10), the fluctuations are scale-invariant and their strength is independent of temperature, being instead controlled by the numerical parameter N/q^2 and decreasing/increasing with increasing N and q, respectively. As opposed to the fluctuations about the mean-field solution (6), those associated with the one given by $g_0(\tau)$ correspond to the pseudo-Goldstone excitations about the fixed 'valley' in the space of field configurations, which no longer needs to be adjusted.

Another uniquely simple (and previously unexplored) situation is the case of the Toda potential which, upon a global anisotropic coordinate rescaling, reduces to $V_M(g) = J^2 \cosh 2g$ and coincides with its second derivative up to a factor. Its classical equation of motion assumes the form of the celebrated sinh-Gordon equation, $\partial_\tau^2 g = -J^2 \sinh g$, the solution of which satisfying the initial condition $g(0) = 0$ reads

$$g_0(\tau) = -\ln \tan\left(J\tau + \frac{\pi}{4}\right). \qquad (26)$$

Other known (quasi)solvable potentials are likely to provide novel mean-field solutions, alongside the associated actions for their fluctuations.

A complete boundary theory might also contain additional matter fields, such as an additional $U(1)$ scalar field in the case of the charged (Dirac instead of Majorana) SYK model.

5. Particle in magnetic field

The Hamiltonians akin to those in the previous section routinely arise in the problem of a non-relativistic particle subject to a certain 2d static geometry $g_{ij}(x, y)$ and a vector potential $A_i(x, y)$. By exploiting this analogy one can then replace a field-theoretical path integral over the fluctuating variable $\phi(\tau)$ with a worldline one governed by the single-particle action

$$S[X]=\int \mathrm{d}\tau\left(\frac{1}{2}g_{ij}\partial_\tau X^i\partial_\tau X^j+\partial_\tau X_i A^i\right), \quad (27)$$

where $X_\mu = (x, y)$. This equivalence is limited to the contributions of all single-valued (non-self-intersecting) curves which indeed dominate for low temperatures.

In the hyperbolic plane (H^2) geometry such a connection between the 'particle-in-magnetic-field' (PMF) problem and the SYK model has been utilized before [38–66]. It can be further extended towards a broader class of metrics and magnetic field configurations. As a technical simplification one can first explore the class of diagonal bulk metrics, $g_{ij}(x, y) = \mathrm{diag}[g_{xx}(x), g_{yy}(x)]$, and vector potentials in the Landau gauge, $A_i(x, y) = (0, A_y(x))$, with these choices facilitating a separation of variables in the corresponding Schrödinger equation with the Hamiltonian

$$H_{\mathrm{PMF}}=\frac{1}{2}g^{xx}\pi_x^2+\frac{1}{2}g^{yy}(\pi_y-A_y)^2, \quad (28)$$

where π_i is the conjugate momentum.

For the sake of the following discussion, the background fields can be further restricted to the power-law functions of the x coordinate (here below l is a characteristic length scale akin to the 'AdS radius')

$$g^{xx} = (x/l)^{2\alpha},\ g^{yy} = (x/l)^{2\beta},$$

$$A_x = 0,\ A_y = Bl(l/x)^\gamma V, \quad (29)$$

so that the interval in this (Euclidean and, in general, anisotropic) metric reads $\mathrm{d}s^2 = l^{2\alpha}\mathrm{d}x^2/x^{2\alpha} + l^{2\beta}\mathrm{d}y^2/x^{2\beta}$.

In general, the Hamiltonian dynamics described by Eq. (29) develops in the 4d phase space spanned by two pairs of canonically conjugated variables, (x, π_x) and (y, π_y). However, in the above gauge the y variable becomes cyclic and the conjugate momentum $\pi_y = k$ is conserved, as in a translationally-invariant plane wave-like solution propagating along the 1d boundary of a 2d region.

By comparison, the y variable can be parallelled with the aforementioned 'center-of-mass' time T. In contrast, dynamics in the x direction remains non-trivial and is analogous to the dependence on the 'relative' time τ.

The magnetic flux through the semi-space $x \geq 0$,

$$\Phi=\int \mathrm{d}x\mathrm{d}y\sqrt{g}(\partial_x A_y-\partial_y A_x)=B\int\frac{\mathrm{d}x\mathrm{d}y}{y^{\gamma+1-\alpha-\beta}}, \quad (30)$$

scales with the area provided that $\gamma + 1 = \alpha + \beta$.

A uniform magnetic field in the flat space corresponds to $\alpha = \beta = 0$ and $\gamma = -1$, while its much-studied counterpart on a hyperbolic plane H^2 can be attained for $\alpha = \beta = \gamma = 1$.

Quantizing the PMF Hamiltonian (28) and factorizing its eigenstates, $\Psi(x, y) = \psi(x)\mathrm{e}^{\mathrm{i}ky}$, one arrives at the Schrödinger equation with the quasi-1d Hamiltonian

$$H=\frac{1}{2}x^{2\alpha}\pi_x^2+$$
$$+\frac{1}{2}(x^{2\beta}k^2-2x^{2\beta-\gamma}B\pi_x+B^2x^{2\beta-2\gamma}), \quad (31)$$

which contains a triad of terms governed by the exponents 2β, $2\beta - \gamma$ and $2\beta - 2\gamma$.

Moreover, the Hamiltonian (31) could acquire still higher powers of x stemming from the relativistic corrections proportional to $(\pi_i - A_i)^{2n}$ with $n > 1$.

For $\alpha = 1$ and with the use of a logarithmic reparametrization $x = \mathrm{e}^z$ one can cast Eq. (31) in the form of the ordinary 1d Schrödinger equation in the flat space with the potential $V(z)$ given by Eq. (13). Incidentally, the metric takes the form $\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{e}^{-2\beta z}\mathrm{d}y^2$.

In contrast, for $\alpha \neq 1$ the corresponding 2nd order differential equation would exhibit the power-law potential $V(z) = \sum_n c_n z^n$ after the reparametrization $z = x^{1-\alpha}/(1 - \alpha)$ and rescaling $y \to y(1 - \alpha)^{-\beta/(1-\alpha)}$, in which coordinates the metric takes the form $\mathrm{d}s^2 = \mathrm{d}z^2 + z^{-2\beta/(1-\alpha)}\mathrm{d}y^2$.

Moreover, for $\alpha = 1$ and non-zero k and B the three-term potential in Eq. (31) reduces to only two terms, provided that the other two exponents are related as $\beta = 0$, $\beta = \gamma$, or $\beta = \gamma/2$.

In the first two cases, one obtains the Morse potential (14) with $\lambda = kl/2\gamma$ and $\lambda = Bl^2/2\gamma$, respectively. Thus, the Morse scenario extends beyond the well-known case of a constant field and the H^2 space of a constant negative curvature for $\alpha = \beta = \gamma = 1$. Nonetheless, the magnetic flux Φ can only be proportional to the area $\int \mathrm{d}x\mathrm{d}y$ for $\beta = \gamma$, but not in the other two cases.

By contrast, the third combination of the parameters yields the Toda potential (17) with $c_q = B^2l^4$ and $c_{-q} = k^2l^2$, which conforms to the symmetric potential $\cosh 2\alpha z$ upon a uniform re-scaling $z \to z + \frac{1}{2}\ln(kl/B)$.

For a given PMF Hamiltonian much information can be inferred from its resolvent, which allows for a spectral expansion over its 2d eigenstates

$$D_E(x,y\,|\,x',y') = \langle x,y\,|\,\frac{1}{E-H+\mathrm{i}0}\,|\,x',y'\rangle = \int \mathrm{d}k \sum_{n/v} \mathrm{e}^{\mathrm{i}k(x-x')} \frac{\psi_{k,v}(y)\psi^*_{k,v}(y')}{E-E_{k,v}+\mathrm{i}0}, \quad (32)$$

where $\cosh d = 1 + ((x - x')^2 + (y - y')^2)/2xx'$ and the sum/integral $\Sigma_{n/v}$ is over the discrete and/or continuous parts of the spectrum.

In the case of the 1d Morse potential, Eq. (32) can be computed in a closed form

$$D_E(x,y\,|\,x',y') \sim \left(\cosh\frac{d}{2}\right)^{2\mathrm{i}v-1} \frac{\Gamma(1/2+\lambda-\mathrm{i}v)\Gamma(1/2-\lambda-\mathrm{i}v)}{\Gamma(1-2\mathrm{i}v)} \times F\left(\frac{1}{2}+b-\mathrm{i}v, 1-2\mathrm{i}v, \frac{1}{\cosh^2 d/2}\right), \quad (33)$$

where F is the hypergeometric function.

Fourier transforming (33), one obtains a fundamental solution for the Morse potential for positive

$$d_{E,k}(y,y') = \int \mathrm{d}x \mathrm{e}^{\mathrm{i}k(y-y')} D_E(x,y\,|\,x',y') \sim \frac{\Gamma(1/2-\lambda-\mathrm{i}\sqrt{E})}{k\Gamma(1-2\mathrm{i}\sqrt{E})} \times \sqrt{xx'} W_{\lambda,\mathrm{i}\sqrt{E}}(kx_<) M_{\lambda,-\mathrm{i}\sqrt{E}}(kx'_>) \quad (34)$$

(here $E = v^2 + 1/4 + \lambda^2$), as well as negative

$$d_{E,k}(x,x') \sim \sqrt{xx'} K_{\mathrm{i}v}(kx_>) I_{\mathrm{i}v}(kx_<) \quad (35)$$

energies.

In the zero-field limit, $v = \sqrt{E - 1/4}$ and Eqs. (34) and (35) reduce to

$$D_E(x,y|x',y') \sim Q_{-1/2+iv}(\cosh d) \quad (36)$$

and

$$d_{E,k}(y,y') \sim I_{-\mathrm{i}\sqrt{2E}}(kx) K_{\mathrm{i}\sqrt{2E}}(kx'), \quad (37)$$

respectively.

Also, in the flat space limit, $kl \to \infty$, Eq. (36) reproduces the well-known results

$$D_E(x,y\,|\,x',y') = \frac{\Gamma\left(\frac{B-4E}{2B}\right)}{\pi\sqrt{Br^2/2}} W_{2E/B,0}(Br^2/2) \quad (38)$$

and $E_n = b(2n + 1)$ for the degenerate Landau levels, as all the scattering states are pushed to infinity.

Another important calculable is the thermodynamic propagator (‘heat kernel’)

$$K_\beta(x,y|x',y') = \langle x,y\,|\,\mathrm{e}^{-\beta H}\,|\,x',y'\rangle = \int \mathrm{d}k \sum_{n/v} \mathrm{e}^{\mathrm{i}k(x-x')-\beta E_{k,n/v}} \psi_{k,v}(y)\psi_{k,v}(y'). \quad (39)$$

At zero field (i.e. in the case of the Liouville potential) it simplifies to

$$K_\beta(x,y\,|\,x',y') \sim \exp\left(-\#\frac{\beta}{l^2} - \frac{r^2}{\beta}\right)\frac{l}{\beta}\sqrt{\frac{r}{l\sinh r/l}} \quad (40)$$

and can be used for studying the system’s thermodynamic properties.

6. Thermodynamics and chaos

A partition function for the (generalized) SYK action given by Eqs. (8) and (13) is given by the field-theoretical path integral

$$Z_{\mathrm{SYK}}(\beta) = \int \mathrm{d}\phi \int_{\phi(0)=\phi}^{\phi(\beta)=\phi} \mathrm{D}\phi(\tau) \mathrm{e}^{-\int_\tau S_{\mathrm{SYK}}[\phi(\tau)]}. \quad (41)$$

Alternatively, it can be computed in terms of the eigenfunctions/values $\psi_{n/v}(\phi)$ and $E_{n/v}$ of the corresponding 1d Schrödinger equation

$$Z_{\mathrm{SYK}}(\beta) = \int \mathrm{d}\phi \sum_{n/v} |\,\psi_{n/v}(\phi)\,|^2\, \mathrm{e}^{-\beta E_{n/v}}. \quad (42)$$

Likewise, the PMF partition function is represented by the world-line path integral

$$Z_{\mathrm{PMF}}(\beta) = \int \mathrm{d}x\mathrm{d}y \int_{x,y}^{x,y} \mathrm{D}x(\tau)\mathrm{D}y(\tau) \mathrm{e}^{-\int_\tau S_{\mathrm{PMF}}[x(\tau),y(\tau)]}, \quad (43)$$

where S_{PMF} is constructed from the same Hamiltonian (28). With the use of the eigenfunctions $\Psi_{k,\,n/v}(x, y) = \psi_{k,\,n/v}(x)e^{iky}$ it can be cast in the form similar to Eq. (42),

$$Z_{\mathrm{PMF}}(\beta) = = \int \mathrm{d}x\mathrm{d}y \sum_{n/v} \int \mathrm{d}k \,|\, \Psi_{k,n/v}(x,y)|^2 \, \mathrm{e}^{-\beta E_{k,n/v}}, \quad (44)$$

thus establishing a (pseudo)holographic equivalence between the (generalized) SYK and PMF problems.

Alternatively, instead of performing a direct spectral summation the partition function can be deduced from the density of states (DOS), $Z(\beta) = \int_0^\infty \mathrm{d}E\rho(E)\mathrm{e}^{-\beta E}$.

In turn, the (many-body) DOS of the SYK-like system can be read off from its single-particle PMF counterpart (32):

$$\rho(E) = \frac{1}{2\pi} \mathrm{Im}\, D_E(x, y \,|\, x, y). \quad (45)$$

In the Morse case, using the exact resolvent Eq. (33) one obtains the DOS in a closed form [118–125]

$$\rho_{\mathrm{M}}(E) \sim \frac{\sinh 2\pi\sqrt{E}}{\cosh 2\pi\sqrt{E} + \cos 2\pi\lambda}. \quad (46)$$

For $\lambda = 0$, one then finds the well-known low-energy behaviour of the DOS in the SYK model $\rho(E)$ sin $\sqrt{E}$ [56–66]. In contrast, for $\lambda = 1/2$ the DOS diverges as $\rho(E) \sim 1\sqrt{E}$. Notably, this behaviour is reminiscent of that found in the SUSY version of the SYK model [38–55]. On the other hand, a periodic dependence on λ could be spurious and remains to be better understood.

For $\lambda = 0$, by performing an (inverse) Laplace transformation on Eq. (46) one reproduces the low temperatures partition function of the Liouville model $Z_{\mathrm{L}}(\beta) \sim \exp(O(l^2/\beta))/\beta^{3/2}$ for $\beta J \gg 1$, while for $\beta J \ll 1$ it yields $Z_{\mathrm{L}}(\beta) \sim \exp(O(l^2/\beta))/\beta$. Thus, the specific heat defined as $C = \beta^2\partial_\beta^{\,2}\ln Z(\beta)$ decreases with increasing temperature from $C = 3/2$ down to $C = 1$.

In contrast, the thermodynamic properties of the Morse model appear to be markedly different. Namely, for $\lambda = 1/2$ the specific heat rises from $C = 1/2$ for $\beta J \gg 1$ to $C = 1$ for $\beta J \ll 1$. Together with the aforementioned behaviour of the density of states this might be suggestive of a phase transition at $\lambda_c = 2$.

Such a conductor-to-insulator transition in the SYK double-dot system has been studied, both, without [91–102] and with [89, 90] such a realistic factor as Coulomb blockade taken into account. Conceivably, it is a bulk counterpart of the transition predicted to occur between the SYK non-Fermi liquid and disordered Fermi liquid in a granular array of randomly SYK_2-coupled SYK_4 clusters [126–141].

A difference between the states on the opposite sides of this purported transition can be elucidated with the use of out-of-time-order correlators (OTOC). Generically, the OTOC amplitudes are expected to demonstrate some initial short-time/high temperature exponential growth

$$\frac{\langle G_f(\tau_1,\tau_3)G_f(\tau_2,\tau_4)\rangle}{\langle G_f(\beta/2,0)\rangle^2} = 1 - O\left(\frac{\beta J}{N}\right)\mathrm{e}^{\lambda_{\mathrm{L}} t}, \quad (47)$$

revealed by summing the ‘causal’ ladder series and controlled by the chaotic Lyapunov exponent λ_{L}.

The latter can be deduced directly from Eq. (11) for a general potential $V_{a,\,b}$ upon restoring a dependence of the fluctuating normal mode $\delta g(\tau, T) \sim \mathrm{e}^{\lambda_{\mathrm{L}} T}\chi(\tau)$ on the ‘center-of-mass’ time T and then continuing it analytically, $\tau \to \mathrm{i}t + \beta/2$ [38–66].

This way, one arrives at the eigenvalue equation in terms of the variable $u = \tau/\beta$

$$[-\partial_u^2 + W(g_0(u/\beta))]\chi = -\left(\frac{\lambda_{\mathrm{L}}\beta}{2\pi}\right)^2 \chi, \quad (48)$$

where $W(g(\tau))$ was defined after Eq. (11).

In the case of the Morse potential, one obtains the equation

$$-\partial_u^2\chi - \left(\frac{\cos\theta}{\cosh u + \cos\theta} + \frac{2\sin^2\theta}{(\cosh u + \cos\theta)^2}\right)\chi = = -\left(\frac{\lambda_{\mathrm{L}}\beta}{2\pi}\right)^2 \chi, \quad (49)$$

where the effective potential crosses over from $W_q = 2/\cosh^2 u$ in the pure SYK_q limit ($\theta \to \pi/2$) with the ground state $\chi_q \sim 1/\cosh u$ to $W_{q/2} = 1/2 \cosh^2(u/2)$ in the pure $\mathrm{SYK}_{q/2}$ one ($\theta \to 0$). In both limits, the Lyapunov exponent approaches its maximal value $\lambda_{\mathrm{L}}^{\max} = 2\pi/\beta$ [38–55] as

$\lambda_L/\lambda_L^{max} = 1 - O(\min[\Gamma^2\beta/J, J/\Gamma^2\beta])$ for high/low temperatures [109–112].

In the special case of $q = 4$, though, the fixed-point SYK_2 behaviour corresponds to the disordered but non-chaotic Fermi liquid where λ_L is expected to vanish.

In the intermediate regime and for $q > 4$, the exponent appears to take lower, yet non-zero, values [107, 108]. It does not vanish at any finite temperature, though, thus calling for a closer look at any scenario of a genuine finite-temperature phase transition – or a zero-temperature one predicted to occur at a critical ratio Γ/J vanishing at large N as a power of $1/N$ [100–102]. In that regard, it would be particularly interesting to compute λ_L at the super-symmetric point $J = 2\Gamma$.

In the aforementioned 'variable scaling' model [109–112], some non-maximal and non-universal, yet temperature-independent and growing with the increasing integer parameter n, values of λ_L were reported on the basis of the numerical solution of Ref. (48). In turn, the Hulten potential falls somewhere in between the 'super-symmetric' point at the SYK_q-$SYK_{q/2}$ model and the 'variable scaling' one.

As an interesting consistency check, the eigenfunction equation for the Toda potential $V_{2,-2}$ evaluated on the solution (26) satisfies the same Eq. (49) apart from the constant shift, $W_T = -2/\cosh^2 u + 1$. This constant shift raises the ground state energy to zero, thus implying $\lambda_L = 0$, consistent with the discrete nature of the spectrum consisting only of the bound states.

7. Dual gravities

In 2d, a powerful gauge invariance under local coordinate diffeomorphisms eliminates any bulk degrees of freedom, thereby making such theories locally quantum trivial in the absence of matter. Such bulk theories appear to be topological and allow for explicit classical solutions, thereby providing natural candidates for testing out the foundations of the holographic principle.

Moreover, the gauge symmetry leaves only one independent metric component (e.g. $g_{01} = g_{10} = 0$, $g_{00} = 1/g_{11}$), thus reducing (up to a conformal factor) all the (Euclidean) metrics to the set $ds^2 = e^{2\phi(x)}d\tau^2 + e^{-2\phi(x)}dx^2$ parametrized by a single function ϕ.

However, a 2d gravity theory can still develop a non-trivial boundary behaviour as a result of introducing either an additional dilaton, Liouville, or scalar matter field. Alternatively, it requires anisotropic space vs time scaling, $t \sim x^z$, characterized by a dynamical critical index z. Thus, such extensions can be sought out not only in the context of generalized JT but also the Horava–Lifshitz (HL) [142–144] theories.

The original (ostensibly 2d) JT model is well known to be described by the Schwarzian boundary action providing a natural holographic connection to the edge modes propagating along the 1d boundary [67–88]. Indeed, the Schwarzian can be directly related to the extrinsic curvature of a fluctuating closed 1d boundary of a 2d region, $K = 1 + Sch\left\{\tan\frac{\pi f}{\beta}, \tau\right\} + \ldots$. Therefore, from the formal mathematical standpoint Eq. (8) represents the action of the Virasoro group on its coadjoint orbits.

In practical terms, establishing a generalized holographic duality with a given Liouville-type theory described by Eqs. (8) and (13) can be formulated as a task of constructing the bulk theory, the boundary dynamics of which is governed by the same 1d Hamiltonian as that of the purportedly dual 1d quantum system.

In that regard, the boundary actions of SUSY and higher spin extensions of the 2d dilaton gravities were argued to represent certain specific limits of the generalized JT model, including its non- and ultra-relativistic variants [67–88]. Alternatively, the complex SYK model was argued to have a possible flat space bulk dual.

The most complete action incorporating a dynamical dilaton reads

$$S = \int dx d\tau \sqrt{g}[R\Phi + U(\Phi)(\partial\Phi)^2 + V(\Phi)], \quad (50)$$

parametrized in terms of the functions U and V of the dilaton field Φ. Such generalized dilaton gravities have been encountered among the deformations and compactifications of higher-dimensional theories. Among them, there is an important two-parameter family of potentials $U = a\Phi$ and $V = b\Phi^{a+b}$, among which the one-parameter subset with $a + b = 1$ possesses the AdS_2 ground state [67–88].

Moreover, the so-called $F(R)$-gravities with the action $S_{F(R)} = \int dx dt F(R)$ have been argued to be all equivalent to the 'minimal' JT action with $F(R) = \Phi R - V(\Phi)$, the expectation value of the dilaton field being related to the Ricci curvature, as per the equation $R = \partial V/\partial \Phi$ [145–147].

Alternatively, the intrinsically topological nature of the JT gravity can be made manifest with the use of its 1st order formulation [148–155]

$$S_{\mathrm{JT}} = \int \mathrm{d}\tau \mathrm{d}x [\Phi \in_{\mu\nu} d^{\mu}\omega^{\nu} + \\ +W(\Phi) \in^{ab} \in_{\mu\nu} \mathrm{e}^{\mu}_{a} \mathrm{e}^{\nu}_{b} \\ +X^{a} \in_{\mu\nu} d^{\nu} \mathrm{e}^{\mu}_{a} + X^{a} \in^{b}_{a} \omega_{\mu} e^{\mu}_{b}], \quad (51)$$

in terms of the vielbein e^{μ}_{a} and spin-connection ω^{μ} which is independent of the background metric $g_{\mu\nu} = \eta_{ab} \mathrm{e}^{a}_{\mu} \mathrm{e}^{b}_{\nu}$ (here η is the flat space metric). Notably, the action (51) shares its topological nature with the 3d gravity that can be cast in terms of the (twinned) Chern–Simons theory [156–161].

Another viable candidate to the role of a 2d gravity dual to a generalized SYK model can be sought out in the form of the (Lorentz non-invariant) Horava–Lifshitz action [142–144]

$$S = \int \mathrm{d}x \mathrm{d}\tau \sqrt{g} N(\alpha K^2 + b\Lambda + c(N'/N)^2), \quad (52)$$

where a, b and c are numerical parameters, N and N_x are the lapse and shift functions, $h = \sqrt{g_{xx}}$, and $K = -(\dot{h}/h - \dot{N}'_x/h^2 + N_x h'/h^3)/N$, the dots and primes standing for the time and space derivatives, respectively. In contrast to the dilaton gravity (51) Eq. (53) is only invariant under the foliation-preserving diffeomorphisms $\tau \to \tau'(\tau)$ and $x \to x'(x, \tau)$.

Certain previously proposed $F(R)$-HL theories provide the Lifshitz-type black hole solutions with a constant negative curvature $R = -2z^2/l^2$.

Under the projectability condition [142–144] one can choose $N = N(\tau)$ to be a global (spatially uniform) variable which then gives rise to $K' = 0$ (hence, $K = K(\tau)$). Furthermore, by using the coordinate gauge symmetry one can fix $N = 1$ and $N_x = 0$. The number of primary and secondary Hamiltonian constraints equals the dimension of the phase space, thus reducing the number of dynamical bulk degrees of freedom to zero. This observation suggests that the conjugate momentum $p = p(\tau)$ is independent of the spatial position either.

Canonically quantizing the action (52) one then arrives at the effectively 1d PMF-like Hamiltonian

$$H_{\mathrm{HL}} = aqp^2 + b\Lambda q + c\frac{P}{q^{w}}, \quad (53)$$

where $q(\tau) = \int \mathrm{d}x h(x, \tau)$ and $p(\tau)$ constitute a pair of conjugate canonical variables while the 2nd variable $Q(\tau)$ is cyclic and paired up with a conserved conjugate momentum $P(\tau) = P$.

In the context of Friedmann–Robertson–Walker (FRW) cosmology, the Hamiltonian (53) emerges in the Wheeler–DeWitt equation, the parameter w taking values 1, 0 and −1 for radiation, matter, and dark energy, respectively. Contrasting Eq. (53) against Eq. (31) one finds that the essential terms in the two expressions match for, e.g. $\alpha = 1/2$, $\beta = -w - 1/2$ and $\gamma = -1 - w$.

Adding matter to Eq. (53) introduces another pair of conjugate variables, similar to the formulation of the PMF problem in a non-separable gauge. In particular, the projectable HL action with an additional scalar field $Q(x, \tau) = Q(\tau)$ governed by a potential $V(Q)$ and paired with a conjugate momentum $P(\tau)$ reduces to the generic 2d PMF Hamiltonian [142–144]

$$H_{\mathrm{HL+S}} = aqp^2 + b\Lambda q + c\frac{P^2}{q} + qV(Q). \quad (54)$$

8. Summary

The orthodox holographic scenario requires a bulk gravity to have non-trivial dynamics that gets quenched and turns classical only in a certain ('large-N') limit [20–26].

In that regard, the SYK-JT duality would often be referred to as a genuine case of low-dimensional holographic correspondence. It is generally agreed, though, that such equivalence does not quite rise to the level of the aforementioned full-fledged holographic duality, as the JT bulk dual is non-dynamical and determined by the boundary degrees of freedom, thus making both systems effectively 1d.

This note argues that similar (pseudo)holographic relationships can be established between the various extensions of the original SYK model and more general (JT, $F(R)$-, HL, etc.) 2d gravities. The correspondence between their low-energy sectors presents a form of equivalence between different realizations of the co-adjoint orbits of the (chiral) Virasoro group.

Formally, both sides of such duality can be described in terms of some 1d Liouvillean quantum mechanics, thus generalizing the pure Schwarzian action, the description of which can also be mapped onto an equivalent (single particle) PMF problem.

From the practical standpoint, certain analytically solvable quantum-mechanical potentials can then be related to the physically relevant SYK deformations, such as the action given by Eqs. (8) and (14) for a double SYK quantum dot.

The PMF analog picture allows for direct access to the resolvent D_E and heat kernel K_β functions, thus allowing one to compute the density of states $\rho(E)$, partition function $Z(\beta)$, and other thermodynamic properties of the boundary SYK-like system of interest. By further utilizing this approach one can also study various quantifiers of entanglement, quantum chaos, and even more subtle $n \geq 2$-body correlations.

Furthermore, the tangle of (pseudo)holographic relationships between the $SL(2, R)$-symmetric boundary (Schwarzian/Liouville-like) and bulk (JT/HL-like) models can be viewed as different forms of embedding (at fixed radial and angular vs temporal and angular coordinates, respectively) into the global AdS_3 space [113–117]. Importantly, a similar relationship also exists between the 1 + 2-dimensional gravity with its Banados–Teitelboim–Zanelli black hole backgrounds and various (e.g. Korteweg-de Vries) families of solvable 1 + 1-dimensional quantum systems [156–161]. Among other things, such equivalence can be utilized to study nonlinear hydrodynamics of the soliton-like edge states of generalized bulk QHE systems [162–188].

Thus, when seeking out genuine implementations of the central holographic 'IT from QUBIT' paradigm one might first want to make sure that the conjectured duality does not appear to be of the 'ALL from HALL' variety. Indeed, discovering a (possibly, hidden) topological origin of holographic correspondence would greatly help to demystify this otherwise fascinating, yet baffling, concept.

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IT IŠ KUBITO AR VISKAS IŠ HOLO?

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