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THE *GLORIA MUNDI* OF SYK DOES NOT TRANSIT YET

D.V. Khveshchenko

Department of Physics and Astronomy, University of North Carolina, Chapel Hill, NC 27599, U. S. A.
Email: khvesh@physics.unc.edu

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This paper discusses the examples of 0 + 1-dimensional Liouvillean dynamics instigated by the various deformations of the Sachdev–Ye–Kitaev (SYK) model. In reference to such deformations the main focus is on the regions of parameter space where the competing SYK couplings are of a comparable strength and cannot be treated as each other's perturbations in the vicinity of the conformal fixed points corresponding to the pure SYK_q models with different values of q. Crossovers between such fixed points ('SYK transits') can be efficiently studied in the equivalent framework of single-particle quantum mechanics.

Keywords: SYK model, Liouvillean quantum mechanics, Lyapounov exponent

1. The rise of SYK

The glorious rise of the celebrated SYK model [1–8] into one of the central themes in modern interdisciplinary theoretical studies was due to a rare confluence of such precious properties as its elegant solubility, maximally chaotic behaviour, asymptotic conformal symmetry, and more. The numerous in-depth analyses of the SYK model revealed a number of important connections between such seemingly disjoint subjects as random matrices, quantum black holes, disordered quantum dots, and, possibly, strange metallic behaviours in the various condensed matter systems.

One of these novel connections may have already contributed towards a resolution of the long-standing black hole information paradox by demonstrating that the properly (re)defined Hawking radiation entropy can be unitary, following the previously predicted Page curve [9, 10].

In the condensed matter context, the SYK model has served as a powerful inspiration for a great many proposed non-Fermi-liquid (NFL) scenarios [11–26]. However, the very existence of numerous plausible explanations of, e.g. the ubiquitous linear temperature dependence of resistivity in bad metals [27–43], may seem to suggest that its ultimate explanation is yet to be found.

Nevertheless, alongside the renewed interest in hydrodynamics inspired by the holographic ideas, the SYK scenaria have been particularly important for pursuing the *ad hoc* field of 'bottom-up' (a.k.a. 'non-AdS/non-CFT') holography which purports to describe a variety of (allegedly) strongly correlated condensed matter systems [44–50]. Indeed, with the once abundant and defiantly upbeat claims of 'explaining' high-T_c materials, heavy fermions, graphene, etc. by virtue of some uncontrolled calculations in the conveniently chosen (and/or previously studied) classical gravity theories all but gone, the SYK model has remained a rather unique theoretical playground for obtaining rigorous results.

In that regard, the SYK model would be often referred to as a genuine example of low-dimensional holographic correspondence – even despite the fact that, both being effectively one-dimensional, the low-energy sector of SYK and its dual (formally, two-dimensional) Jackiw–Teitelboim (JT) gravity present a form of equivalence between different realizations of the quantized co-adjoint orbit of the (chiral) Virasoro group. Such equivalence does not quite rise to the level of full-fledged holographic duality, as the JT bulk dual is non-dynamical and determined by the boundary degrees of freedom. By contrast, in order to qualify as a true holographic scenario the bulk theory would have to have some non-trivial bulk dynamics that gets quenched and turns classical only in a certain ('large-N') limit [44–50].

Moreover, similar remarks can also be made about the (historically, somewhat less extensively discussed) correspondence between the 3d gravity with BTZ-like black hole backgrounds and the various (KdV and alike) families of solvable 1 + 1-dimensional systems (see, e.g. Ref. [51] and references therein).

2. The SYK deformations

Since the beginning of the SYK era there have been attempts to explore deviations from the original SYK_4 model in order to assess the generality (or, conversely, uniqueness) of the behaviour that it represents. In particular, there has been much discussion of the conjectured NFL–Fermi liquid (FL) transition in the hybrid SYK_4–SYK_2 model [1–8, 11–26].

A renormalization flow between the two fixed points has been mostly studied by means of perturbation theory operating in terms of the propagator $G(\tau_1, \tau_2)$ of $N \gg 1$ spaceless Majorana fermions [11–26]. In the conformal limit of the generalized SYK_q of order $q \geq 4$ the latter exhibits the fermion dimension $\Delta = 1/q$, thus making the perturbation proportional to $G^{q/2}$ strongly relevant (dimension one) in the near vicinity of the UV SYK_q fixed point. Conversely, the formerly leading term G^q becomes strongly irrelevant (dimension four) near the IR fixed point of $SYK_{q/2}$. A unique feature of the $q = 4$ case, though, is that the transition occurs not between two different NFLs but the SYK_4 NFL and the disordered FL.

Notably, in the course of crossing over between the different fixed-point regimes the value of q of the dominant term plays a role akin to that of the central charge in 2d conformal field theories.

As far as the potential physical applications are concerned, some of the previous analyses [11–26, 52–55] suggest that the putative phase transition may take place at critical couplings vanishing as powers of $1/N$ – which value would be practically indistinguishable from zero in a macroscopic system – while others yield values that remain finite in the $N \to \infty$ limit.

The common approach to a SYK-type model starts out by integrating the fermions out, thereby arriving at the action in terms of the bi-local fields $G(\tau_1, \tau_2)$ and the corresponding self-energy $\Sigma(\tau_1, \tau_2)$,

$$S = \frac{N}{2}\mathrm{Tr}\ln(\partial_\tau - \Sigma) + \frac{N}{2}\int\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \left\{(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - F[G(\tau_1,\tau_2)]\right\}, \quad (1)$$

where the functional $F[G]$ results from averaging over the Gaussian-correlated random amplitudes of all-to-all q-body entanglement. Moreover, such entangling couplings can be made non-uniform, thus introducing a notion of spatial dimensions and further extending the class of attainable models to include those with 'distance'-dependent entanglement [56, 57].

Solving for the self-energy, the Schwinger–Dyson equation derived from (1) takes the form

$$\partial_{\tau_1} G(\tau_1,\tau_2) - \int \mathrm{d}\tau_3 \left\langle \tau_1 \left| \frac{\delta F}{\delta G} \right| \tau_3 \right\rangle G(\tau_3,\tau_2) = \delta(\tau_1 - \tau_2). \quad (2)$$

In the original SYK_q model with $F(G) = J^2G^q$ Eq. (2) remains invariant under any diffeomorphisms $\tau \to f(\tau)$ of the thermodynamic time variable τ subject to the boundary condition $f(\tau + \beta) = f(\tau) + \beta$ as long as the derivative term is neglected and provided that G and Σ transform as

$$G(\tau_1, \tau_2) \to G_f = (f'(\tau_1)f'(\tau_2))^{\Delta}G(f(\tau_1), f(\tau_2)),$$

$$\Sigma(\tau_1, \tau_2) \to \Sigma_f = (f'(\tau_1)f'(\tau_2))^{1-\Delta}\Sigma(f(\tau_1), f(\tau_2)). \quad (3)$$

At $J\tau \gg 1$ a representative power-law solution to Eq. (2) reads $G_0(\tau_1, \tau_2) \sim \mathrm{sgn}\tau/(J\tau)^{2\Delta}$ (hereafter $\tau = \tau_1 - \tau_2$).

Choosing a particular mean-field solution reduces the invariance under arbitrary diffeomorphisms down to the subgroup of the Mobius transformations $SL(2, R)$. Correspondingly, a gradient expansion of the logarithm in Eq. (1yields the (approximately) local effective action which describes the finite temperature dynamics of the reparametrization mode [58–79]

$$S_0(f) = -O(1)\frac{N}{Jq^2}\int \mathrm{Sch}\left\{\tan\frac{\pi f}{\beta}, \tau\right\}\mathrm{d}\tau, \quad (4)$$

where Sch stands for the Schwarzian derivative $\mathrm{Sch}\{f,x\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ obeying the differential 'chain rule' $\mathrm{Sch}\{F(f), x\} = \mathrm{Sch}\{F(f), f\} \times$

$\times f'^2$ + Sch{f, x} and operating on the manifold of (nearly) degenerate states related by virtue of the transformations (3).

3. Liouvillean quantum mechanics

Under the customary parametrization $f'(\tau) = e^{\phi(\tau)}$ the Schwarzian action (4) assumes the (pseudo-) free form $S_0(\phi) \sim \int d\tau(\phi')^2$. In the process of averaging the products of propagators

$$\langle G_f(\tau_1,\tau_2)...G_f(\tau_{2p-1},\tau_{2p})\rangle = \\ = \int D\phi e^{-S_0(\phi)} \prod_{i=1}^{p} \frac{e^{\Delta(\phi(\tau_{2i-1})+\phi(\tau_{2i}))}}{(\int_{\tau_{2i-1}}^{\tau_{2i}} d\tau e^{\phi})^{2\Delta}} \tag{5}$$

over the fluctuations of ϕ, the action $S_0(\phi)$ gets augmented by the Liouville term $\Delta S_2(\phi) = h_2\int d\tau e^{2\phi(\tau)}$ with $h_2 \sim J$. Technically, upon promoting the denominator in (5) to the exponent with the help of some auxiliary integration *a la* Feynman the overall effective potential acquires a piece-wise Liouville term acting during the time intervals between $2p$ consecutive insertions of the operator $e^{i\Delta\phi}$ [81, 82].

The resulting action $S_0 + \Delta S_2$ can then be quantized by considering the corresponding (rescaled) Schroedinger equation [81, 82]

$$\left(-\frac{1}{2}\partial_\phi^2 + h_2 e^{2\phi}\right)\psi = E\psi, \tag{6}$$

the scattering states of which $\psi_k(z) \sim K_{2ik}(2\sqrt{z})$ (here $z = \lambda e^{\phi}$) belong to the continuum with the spectrum $E_k = k^2$ and the density of states $\rho(E) \sim \sinh 2\pi\sqrt{2NE/J}$ [58–79, 81, 82].

These exact expressions can be used to compute the matrix elements $\langle 0|e^{\Delta\phi}|k\rangle$ exactly. Such calculations reveal the universal limit of an arbitrary power of G_f averaged over the soft mode fluctuations in the late-time, $\tau > N/J$ – or, at finite temperatures, in the strong coupling, $J\beta/N \gtrsim 1$ – regime [81, 82],

$$\langle G_f^p\rangle \propto \frac{1}{(J\tau)^{3/2}}. \tag{7}$$

This behaviour is markedly different from the (non-universal) mean-field one $G_0^p \propto 1/\tau^{2p/q}$ at shorter times ($\tau < N/J$) or weak couplings ($J\beta/N \lesssim 1$).

In that regard, the presence of the exponential term $\Delta S_2(\phi)$ in the overall effective action is instrumental. In its absence the Gaussian fluctuations of the field ϕ governed by $S_0(\phi)$ would have caused non-algebraic decay, thus being unable to deliver the universal power-law (7). In fact, such a behaviour could have never emerged out of the purely Gaussian ϕ fluctuations even if the correlator $\langle\phi(\tau)\phi(0)\rangle$ were logarithmic, as $\ln\langle G_f^p\rangle$ would still depend on p and Δ (in both cases, quadratically).

The effective action might also include the various intrinsically non-local terms

$$\Delta S_n = h_n \int\int d\tau_1 d\tau_2 \left\{\frac{f'(\tau_1)f'(\tau_2)}{[f(\tau_1)-f(\tau_2)]^2}\right\}^{\Delta_n}, \tag{8}$$

that can dominate over (4) for $\Delta_n \le 3/2$. In the previous analyses, such terms would be routinely substituted with the local operators $\Delta S_n = h_n \int d\tau\, e^{\Delta_n\phi(\tau)}$ thus further modifying the equivalent quantum mechanical Hamiltonian in (6).

4. Bi-quadratic SYK deformation

This important example of the deformed SYK model has been extensively discussed in the context of random tunnelling between two different SYK systems. For example, it arises in such, at first sight, unrelated fields as theoretical cosmology ('traversable wormhole') [58–79] and coupled quantum dots [83–93].

In most analyses, the perturbed propagator would be taken in the form (3) of a 'gauge-transformed' mean-field solution G_0, thereby accounting for the 'soft' reparametrization mode $f(\tau)$ while ignoring any potential changes to the mean-field background field configuration itself.

In particular, adding the SYK_2 ('tunnelling') term with the amplitude Γ replaces the Liouville potential in the SYK_4 action (written in the Euclidean signature) with the Morse-type one [94–97]

$$S = \int d\tau \left[\frac{1}{2J}(\phi')^2 + Je^{2\phi} + \frac{\Gamma^2}{J}e^{\phi} + \frac{1}{\beta^2 J}e^{4\phi}\right]. \tag{9}$$

The corresponding Schroedinger equation with the (properly rescaled) Hamiltonian

$$H = -\frac{1}{2}\partial_\phi^2 + e^{2\phi} + \lambda e^{\phi} + \lambda' e^{4\phi}, \tag{10}$$

that can be solved exactly in terms of the wave functions (here $z = 2\lambda e^{\phi}$) $\psi_k(z) \sim e^{-\phi/2}W_{\lambda,ik}(z)$ with the continuous spectrum $E_k = k^2 + 1/4 + \lambda^2$

if the last – subdominant at low temperatures (or large negative ϕ) – term in (10) can be dropped [94–98].

Besides, for $\lambda < 0$ the Hamiltonian (10) appears to possess a finite number of bound states

$$\psi_n(z) \sim z^{\lambda-n-1/2-z/2} L_n^{2\lambda-2n-1}(z) \tag{11}$$

at the discrete energies $E_n = -(\lambda - n + 1/2)^2$, $n = 0, \ldots, [\lambda - 1/2]$. Near its minimum this spectrum can be approximated by the oscillator one.

A somewhat different path leading up to the Morse-type action (9) was taken in Ref. [98]. The effect of the tunnelling term with $\Delta_1 = 1/2$ was argued to be two-fold: first, it contributes to (and/or refines) the purely Schwarzian (or 'hard' mode) saddle-point solution and, second, controls the pseudo-Goldstone (or 'soft' mode) fluctuations. These roles would be separately played by the 'longitudinal' (or radial, $e^{\xi} = 1 - f'$, in the holographically dual JT picture) and 'transverse' (or angular, ϕ) fluctuations, respectively. The former was argued to be strongly non-Gaussian and the effect of such fluctuations was claimed in Ref. [98] to strengthen (somewhat unexpectedly) the SYK_4 conformal mean-field behaviour over a broader range of parameters.

More specifically, the strong coupling Schwarzian regime was argued to sustain the SYK_2 perturbation at all couplings $\gamma \equiv \Gamma/J$ below $\gamma_c \sim 1/N$ while at its higher values the propagator was found to crossover to the $q = 2$ FL fixed point. This was argued to be suggestive of a zero-temperature phase transition taking place at γ_c, rather than at a much larger value of order $1/N^{1/2}$, as per the naive estimate. Such parametric reduction of γ_c was claimed to manifest a stabilizing effect of the SYK_2 coupling on the mean-field conformal solution against the Schwarzian fluctuations due to the formation of a polaron-like non-perturbative field configuration.

Correspondingly, the earlier perturbative analysis by the same authors revealed that a weak SYK_2 coupling does not alter the Schwarzian asymptotic (7) up to the values of γ of order γ_c [55].

Such observations appear to be generally consistent with those of Refs. [99, 100] that conjectured the existence of a chaotic-integrable transition in the SYK_q–SYK_2 model at finite temperatures. Above the transition temperature the system was found to behave chaotically while below it the chaos-related Lyapunov exponent (see below) dropped to zero, thus hinting at the FL nature of the underlying ground state.

On the technical side, upon, first, introducing two Lagrange multipliers λ and Λ and, then, voluntarily relaxing the corresponding constraints by fixing their mean-field values, Ref. [98] arrived at the effective action

$$S = \int d\tau \left[\frac{1}{2}(\phi')^2 + \Lambda(e^{2\phi} - f') + \lambda(e^{\phi} - \chi) - \frac{1}{(\beta J)^2} e^{4\phi} \right] - \frac{1}{2} \iint d\tau_1 d\tau_2 \frac{\chi(\tau)\chi(\tau')}{|f(\tau_1) - f(\tau_2)|}. \tag{12}$$

In this (perhaps, somewhat excessive) parametrization, the functional integration about the mean-field SYK_4 fixed point factorizes into, first, taking a quantum mechanical expectation value over the exact ground state $\psi_0(\xi)$ of the Hamiltonian (10) and, then, additionally averaging over the Gaussian (perturbative) ϕ fluctuations. In Ref [98], neither mechanism was found to have any profound effect on the correlators, though.

In particular, an arbitrary power of the mean-field propagator would still retain its bare mean-field form provided that the ϕ-fluctuations were controlled by a large parameter λ. Likewise, averaging over the ground state of (10) adds the square of a non-singular expectation value $\langle 0|e^{p\Delta\phi}|0\rangle = \int d\phi e^{p\Delta\phi}\psi^2_0(\phi)$ which does not give rise to any decaying power-law factor either. In that sense, the largely negligible effect of, both, the Gaussian fluctuations and the ground state averaging may indeed be viewed as increased stability of the mean-field regime in the presence of even a small SYK_2 coupling.

It should be noted, though, that under the assumption of $\lambda < 0$ the Morse potential in (10) appears to differ from that of Refs. [52–54] which is strictly repulsive, monotonic ($\lambda > 0$), and lacks any bound states. It might also be concerning that if the potential in (10) were to support any bound states with $E_n < 0$, then the fluctuation-averaged two-point correlator $\langle G_f(\tau)\rangle = \Sigma_n e^{-E_n\tau} N(E_n)$ would be receiving – on top of the universal term (7) that stems from the continuum of scattering states with $E_k > 0$ – a nonunitary (exponential) contribution, the potential divergence of which could only be arrested by the squared matrix element $N(E_n < 0) = |\langle 0|e^{\Delta\phi}|n\rangle|^2$.

Interestingly, for $J = \Gamma$ the aforementioned monotonic and non-monotonic Morse potentials represent two super-partners fitting into one super-symmetric pair $W_\pm(\phi) = V^2 \pm dV/d\phi$ with $V(\phi) \propto e^\phi$. The ground state of the binding potential then takes the form $\psi_0(\phi) \propto \exp(-\int V\,d\phi)$.

Conceivably, the effective action $S(\phi)$ may develop other interesting regimes at the points of still higher symmetry. One such example would be provided by the Hulten potential

$$W(\phi) = \lambda \frac{e^\phi}{1-e^\phi}, \tag{13}$$

first three terms of the expansion of which in powers of e^ϕ coincide with the 'hyper-symmetric' (or 'tri-critical') point $J = \Gamma = 1/\beta$ in (9). Also, the $1/\phi$-behaviour at a small negative ϕ would be similar to that in the Coulomb potential, although the potential (13) features only a finite number ($[\lambda]$) of bound states at $E_n = -[(\lambda^2 - n^2)/2\lambda n)]^2$.

5. Large *q* limit

An alternate approach to the SYK models exploits the large-q approximation, where the propagator is sought out in the form

$$G(\tau) = \frac{1}{2}\operatorname{sgn}\tau\left(1 + \frac{2}{q}g(\tau) + \ldots\right). \tag{14}$$

Higher order corrections in $1/q$ can also be evaluated, albeit at the increasingly prohibitive costs [58–79].

The action in the path integral over the field g then takes the form

$$S(g) = \frac{N}{q^2}\int\int d\tau_1 d\tau_2 \left[\frac{1}{2}\frac{dg}{d\tau_1}\frac{dg}{d\tau_2} + W(g)\right], \tag{15}$$

with the corresponding equation of motion

$$\partial_\tau^2 g = -\frac{\partial W(g)}{\partial q}. \tag{16}$$

Formally solving (16) one obtains the classical trajectory

$$\tau = \int_{g_0}^0 \frac{dg}{\sqrt{W(g_0) - W(g)}}, \tag{17}$$

with the use of which thermodynamics of the system can be studied by putting $\tau = \beta/2$ [52–54]. In particular, the turning point $g_0 < 0$ of the potential can be directly related to the mean-field energy [54]

$$E = \frac{N}{4q^2}\left[\beta W(g_0) - 2^{3/2}\int_{g_0}^0 dg\sqrt{W_0 - W(g)}\right]. \tag{18}$$

As already mentioned, one possible generalization of the bi-quadratic (Schwarzian plus tunnelling) $q = 4$ action to the larger values of q is provided by the SYK_q–$SYK_{q/2}$ functional

$$F|G| = \frac{2^q J^2}{q^2} G^q(\tau_1,\tau_2) + \frac{2^{q/2}\Gamma^2}{q^2} G^{q/2}(\tau_1,\tau_2). \tag{19}$$

The corresponding effective potential

$$W(g) = J^2 e^{2g} + \Gamma^2 e^g \tag{20}$$

allows for the explicit saddle point solution [52–54]

$$g(\tau) = -\ln(1 + \sqrt{J^2 + 4\Gamma^2}\,\tau + \Gamma^2\tau^2) \tag{21}$$

that gives rise to the mean-field propagator

$$G_0(\tau) = \frac{1}{2}\frac{\operatorname{sgn}\tau}{(1 + \sqrt{J^2 + 4\Gamma^2}\,\tau + \Gamma^2\tau^2)^{2/q}}. \tag{22}$$

For future reference, the final-temperature counterpart of (21) reads $g(\tau) = -\ln[(\sqrt{v^2 J^2/\beta^2 + \Gamma^4} \times \cos(2v\tau/\beta - v) + \Gamma^2)(\beta^2/2v^2)]$, where the parameter v is to be determined from the relation $2v^2 = \Gamma^2\beta^2 + \cos v\sqrt{J^2 v^2 \beta^2 + \Gamma^4\beta^4}$ and becomes $v = 1 - O(1/\beta J)$ for $\Gamma = 0$ [52–54].

It is worth noting that in the look-alike equations (10) and (20) the field variables ϕ and g depend on the 'centre-of-mass' (cf. Eq. (8)) and relative times, respectively. Also, unlike the approximate conformal propagator G_0, the expression (22) is UV-finite and naturally regularized at $\tau \sim \min\,[1/J, 1/\Gamma]$. Hence, by contrast with the latter, the saddle-point solution (22) remains applicable at all y, both large and small. Therefore, the fluctuations of $g(\tau)$ describe pseudo-Goldstone excitations about the fixed 'valley' in the space of field configurations which no longer needs to be adjusted.

6. Quadratic fluctuations

Small fluctuations about the mean-field solution (22) are described by the Gaussian action

$$S_2 = \frac{N}{2q^2}\int\int d\tau_1 d\tau_2 \delta g(\tau_1)\frac{\partial^2 S}{\partial g^2}\bigg|_{g_0}\delta g(\tau_2). \quad (23)$$

For a potential $W(g) = \Sigma_n c_n e^{ng}$ these fluctuations δg would then be governed by a functionally similar kernel $\partial^2 W/\partial g^2 = \Sigma_n c_n n(n-1)e^{ng}$. Albeit similar in its appearance to the previously discussed $S(\phi)$, this action is bi-local and cannot be readily used for deriving the Hamiltonian and quantizing it by means of the substitution $g' \to -i\partial/\partial g$.

In contrast to the Schwarzian action (4) the δg fluctuations are scale-invariant and their strength is independent of energy or temperature, being instead controlled by the numerical parameter N/q^2. For a finite q the strength of such fluctuations decreases with increasing N, yet it remains fixed in the double-scaling limit, $N \to \infty$ and $N/q^2 = \text{const}$.

Inverting the Hessian operator evaluated at the saddle point (21) requires one to find the Green's function of the retarded kernel $D(T,\tau) = \langle \delta_g(T + \frac{\tau}{2}) \delta_g(T - \frac{\tau}{2})\rangle = \langle 12|K^{-1}|12\rangle$ that satisfies the equation

$$\int\int d\tau_5 d\tau_6 \left(-\partial_1\partial_2\delta_{15}\delta_{26} + \left\langle 12\left|\frac{\partial^2 W}{\partial g^2}\right|56\right\rangle\right) \times\langle 56|K^{-1}|34\rangle = \delta_{13}\delta_{24} - \delta_{14}\delta_{23}. \quad (24)$$

Upon Fourier transforming with respect to the 'centre-of-mass' time variable T one can use the spectral decomposition

$$D(T,\tau) = \sum_n \int \frac{d\omega}{2\pi} e^{-i\omega T}\frac{\psi_n\left(\frac{\tau}{2}\right)\psi_n^*\left(-\frac{\tau}{2}\right)}{\omega^2 - \omega_n^2 + i0} = \sum_n \frac{e^{-i\omega_n T}}{\omega_n}\psi_n\left(\frac{\tau}{2}\right)\psi_n^*\left(-\frac{\tau}{2}\right) \quad (25)$$

in terms of the eigenfunctions of the equation

$$\left(-\partial_\tau^2 + \frac{\partial^2 W}{\partial g^2}\bigg|_{g_0}\right)\psi_n(\tau) = \omega_n^2\psi_n(\tau). \quad (26)$$

By analogy with the aforementioned averaging over the ϕ-fluctuations the Gaussian average over δg in the vicinity of the saddle point (22) produces the 'Debye–Waller' factor

$$\frac{\langle G^p(\tau)\rangle}{G_0^p(\tau)} = \langle e^{2p\Delta\delta g(\tau)}\rangle = \exp\{2p^2\Delta^2[D(0,\tau) - D(0,0)]\}. \quad (27)$$

Notably, this averaging is to be performed over the entire function δg, thereby making no distinction between the 'angular' and 'radial' modes.

This might be somewhat similar to, e.g. the standard weak-coupling analysis of the two-dimensional nonlinear $O(N)$ σ-model, that seems to emphasize a distinction between the longitudinal and transverse fluctuations of the order parameter (one gapped and N–1 Goldstone modes, respectively). By contrast, the exact solution demonstrates no such difference as the true $O(N)$-symmetric spectrum consists of the N identical gapped modes.

Evaluating (20) on the classical trajectory (21) at zero temperature one finds the effective potential that asymptotically decays at large τ as $\sim 1/\tau^2$ in both cases of large and small γ. The one-dimensional Green's function of the resulting eigenvalue equation

$$\left(-\partial_\tau^2 + \frac{\kappa}{\tau^2} - \omega^2\right)\psi = 0, \quad (28)$$

with $\kappa > -1/4$, can be found in the closed form

$$D_\omega(\tau,\tau') = \frac{\pi}{2i}\sqrt{\tau\tau'}\frac{H_\nu^{(1)}(\omega\tau_>)}{H_\nu^{(1)}(\omega a)} \times\left[H_\nu^{(1)}(\omega\tau)J_\nu(\omega\tau_<) - H_\nu^{(1)}(\omega\tau_<)J_\nu(\omega a)\right], \quad (29)$$

where $\tau_>$ and $\tau_<$ stand for the larger/smaller τ and τ', respectively, $\nu = \sqrt{1/4+\kappa}$, and a is the UV cutoff.

For $\omega = 0$ (29) amounts to the previously derived expression the finite-temperature version of which reads [58–79]

$$D_0(x,x') = \frac{1}{V\pi}\left[1 + \tan x_<\left(\frac{V\pi}{2} + x_<\right)\right] \times\left[1 - \tan x_>\left(\frac{V\pi}{2} - x_>\right)\right], \quad (30)$$

where $x = \pi\tau/\beta$ and $V = \nu + \frac{2}{\pi}\cot \pi\nu/2$.

Expanding the Bessel functions one once again finds only a mild effect of the Gaussian fluctuations (this time around, of δg), as the ensuing reduction of the amplitude $\langle G^p(\tau)\rangle / G_0^p(\tau) = \exp(O(1)\Delta^2 p^2)$ does not alter the mean-field exponent of the power-law decay.

7. Quadratic fluctuations in *g*-space

As an alternative to (26) one can formulate the eigenvalue equation in terms of the g-variable [54]

$$\left(-2\sqrt{W_0 - W}\partial_g \sqrt{W_0 - W}\partial_g + \left.\frac{\partial^2 W}{\partial g^2}\right|_{g_0}\right)\psi_n = \omega_n^2 \psi_n, \qquad (31)$$

where $W_0 = W(g_0)$ without the need to explicitly solve for the classical trajectory $g(\tau)$.

However, a generally non-trivial derivative $\partial_\tau g$ precludes an immediate use of the known solutions such as (11) in the case of, e.g. the Morse potential $W(g)$. Then treating (31) as a generic second-order equation

$$p(x)\partial_x^2\psi + q(x)\partial_x\psi + (E - V)\psi = 0 \qquad (32)$$

and eliminating the linear derivative term one can convert Eq. (31) into the standard Schroedinger equation with the potential $V' = V + \frac{\partial_x^2 Q}{Q}$ in terms of the wavefunction $\chi = \psi Q$ with $Q(x) = \exp(\int dx\, q/2p)$.

Using this equation in the classically accessible domain $g_0 < g < 0$ one can study the system's thermodynamics. For example, in the case of the Hulten potential (13) one obtains non-trivial temperature dependences of energy $E = E_0 - O(J^{4/3}\beta^{1/3})$ and entropy $S = S_0 - O((J\beta)^{4/3})$ that suggest rather peculiar thermodynamic relations.

8. Ladder eigenfunctions and chaos exponents

A chaotic behaviour may develop in the complementary (classically prohibited) regime $g < g_0$. One popular quantifier of chaos is provided by the out-of-time-order correlator (OTOC) given by the averaged amplitude $\langle G_f(\tau_1, \tau_3) G_f(\tau_2, \tau_4)\rangle$ analytically continued from the domain $\tau_4 < \tau_2 < \tau_3 < \tau_1$ to the complex times $\tau_1 = \beta/4 - it/2$, $\tau_2 = -\beta/4 - it/2$, $\tau_3 = it/2$ and $\tau_4 = -\beta/2 + it/2$.

On top of a non-exponential regular part of the zeroth order in $1/N$ the OTOC function demonstrates an exponentially growing first-order correction. In the case of the SYK_q–$SYK_{q/2}$ model it reads

$$OTOC(t) = O\left(\frac{1}{\beta J}\right) - \frac{f(\gamma)}{B} e^{\lambda_L t}, \qquad (33)$$

where $f(0) = O(1)$. It is controlled by the Lyapunov chaos exponent determined by the ladder eigenstate equation constructed out of the Wightman correlators $G_{lr}(t) = G(\tau = it + \beta/2)$ [58–79]. Taking ω_n in Eq. (26) to imaginary values $\omega_n \to i\lambda_L$ yields the exponentially growing ansatz $D(T, t) \sim e^{\lambda_L T}\psi(t)$, where the real-time eigenfunction $\psi(t)$ solves the equation [54]

$$\left(-\partial_x^2 - \frac{\cos\theta}{\cosh x + \cos\theta} - \frac{2\sin^2\theta}{(\cosh x + \cos\theta)^2}\right)\psi = -\left(\frac{\lambda_L \beta}{2\pi v}\right)^2 \psi \qquad (34)$$

with $\theta = \tan^{-1}(v/J\beta\gamma^2)$. Importantly, the potential in Eq. (34) is monotonic and its sign is opposite of that in Eq. (26).

For $\gamma = 0$ this potential is the original SYK's one, $V_0(x) = -2/\cosh^2 x$, that supports no bound states other than the ground one, $\psi_0(x) \sim 1/\cosh x$, with the eigenvalue $E_0 = -(\lambda_L\beta/2\pi v)^2 = -1$ [58–79]. As has been repeatedly pointed out in the literature, this value of the chaotic operator growth is (almost) maximally possible, its reduction at a strong coupling ($J\beta \gg 1$) being solely due to the temperature-dependent factor v,

$$\lambda_L = \frac{2\pi}{\beta}\left[1 - O\left(\frac{1}{\beta J}\right)\right]. \qquad (35)$$

In the complementary weak coupling regime ($J\beta \ll 1$) the chaotic exponent is $\lambda_L \sim J$.

In principle, the rest of the spectrum in Eq. (34) could provide for some slower growing terms. However, for $\gamma = 0$ no such terms appear as the next (single-node, hence first excited) state would be given by the function $\psi_1 \sim g'$ with the eigenvalue $E_1 = 0$ [54].

Also, at longer times $t \gg \beta$ the behaviour of the OTOC function is determined by the 2-particle density of states, resulting in another universal power-law, $OTOC(t) \propto 1/t^6$ [81, 82].

In Ref. [98], a weak SYK_2 term in (20) was found not to drastically alter the strong-coupling behaviour, except for a reduction of the amplitude by a factor $O(1/J\beta_\gamma^2) < 1$ in the entire interval $1/N \lesssim \gamma \lesssim 1/N^{1/2}$. At such parameter values the Schwarzian fluctuations were found to be suppressed, thus extending the validity of the SYK_4 mean-field solution beyond the energy scale J/N all the way down to $J\gamma^2$ at which the FL behaviour finally sets in.

In Ref. [54], the chaotic exponent of the large-q biquadratic model was computed with the use of perturbation theory about the state ψ_0 for a small γ, thereby finding

$$\lambda_{\mathrm{L}} = \frac{2\pi}{\beta}\left\{1 - O\left[\min\left(J\beta\gamma^2, \frac{1}{J\beta\gamma^2}\right)\right]\right\}. \quad (36)$$

For comparison, Refs. [99, 100] found the exponent $\lambda_{\mathrm{L}} = \frac{2\pi}{\beta}[1 - O(\beta^2\Gamma^2)]$ in the SYK_q– SYK_2 model, suggesting the possibility of a finite-temperature transition for an arbitrarily small Γ.

The latter should, however, be contrasted against the result of Ref. [101] which reported $\lambda_{\mathrm{L}} \sim 1/J\beta^2\gamma^3$ for $\gamma \gg \max\,[1, 1/J\beta]$. Such a behaviour conforms to the generic quadratic temperature dependence of λ_{L} in the disordered FL and could indicate the absence of a genuine finite-temperature phase transition for a sufficiently large Γ.

Adding to the list of possibilities, in Ref. [54] some non-maximal (temperature-independent and growing with the increasing integer parameter n) values of λ_{L} were reported on the basis of a numerical solution of some other ('variable scaling') model with $W(g) \propto 1/(-g)^n$.

As far as more general potentials $W(g)$ are concerned, the Hulten potential (13), for one, falls somewhere in between the 'super-symmetric' ($\gamma = 1$) point of the SYK_q–$SYK_{q/2}$ model and the 'variable scaling' one. The corresponding eigenvalue equation now reads

$$\left[-\partial_x^2 - \frac{2}{\delta}\left(\frac{1}{\cosh x} - \frac{1}{\cosh x + \delta}\right)\right]\psi = -\left(\frac{\lambda_{\mathrm{L}}\beta}{2\pi v}\right)^2\psi, \quad (37)$$

where $\delta = \sqrt{1+4\gamma^2}\,/\,J\beta\gamma^2$. At the super-symmetric point, where $\delta = 5^{1/2}/J\beta$ and for low temperatures ($\delta \ll 1$), the potential in Eq. (37) approaches the original SYK's $V_0(x)$ and the maximally chaotic behaviour $\lambda_{\mathrm{L}} \to \frac{2\pi}{\beta v}$ is once again restored. In the opposite limit of $\delta \gg 1$, the potential flattens out and the Lyapunov exponent decreases monotonically all the way to zero. It does not vanish at any finite temperature, though, thus calling for a closer look at any scenario of a finite-temperature phase transition – or a zero-temperature one predicted to occur at a critical γ_c vanishing as a power of $1/N$.

9. Summary

This paper discussed various generalizations of the SYK model that lead to the one-dimensional Liouvillean quantum mechanics. Of a particular interest are the crossovers between the different conformal fixed points where all pertinent coupling constants are likely to be of the same order. Such 'SYK transits' are not directly amenable to perturbation theory in the vicinity of the fixed points in question but can still be explored in the large-q limit. To that end, one can utilize the already available – and seek out new – non-perturbative mean-field solutions akin to (22) that interpolate between the distinct conformal regimes. This way one could advance the previous studies of the bi-quadratic model (20) and its further extensions within a broader class of the effective potentials $W(g)$.

In particular, this preliminary analysis finds that the Lyapunov exponent at the 'super-symmetric' point of the model (20) remains non-zero down to the lowest temperatures. This observation may call for inspection of the earlier conclusions about the onset of the non-chaotic FL phase at a critical coupling γ_c which could be as weak as $O(1/N^{1/2})$ or even $O(1/N)$ [11–26, 98–100].

Also, further generalizations of the standard Liovillean action related to the various analytically solvable quantum mechanical Hamiltonians might be of interest well above and beyond the original SYK context.

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SYK MODELIO *GLORIA MUNDI* DAR NEPRAĖJO

D.V. Khveshchenko

Šiaurės Karolinos universitetas, Čepel Hilas, JAV