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ON THE USE OF PHOTOELASTIC EFFECT AND PLANE STRAIN OR PLANE STRESS APPROXIMATIONS FOR THE DESCRIPTION OF THERMAL LENSING

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Correct use of the photoelastic effect for the description of thermally induced refractive index change is discussed and the analytical relations between thermo-optic coefficients at zero stresses and zero strains are found for all classes of cubic crystals. These relations may be useful for the investigation of thermal effects in very promising sesquioxide class m3 laser crystals. An accepted set of elasto-optical coefficients of the YAG crystal and an alternative one found in the literature were used in numerical simulations. Significant differences in the calculated thermo-optic coefficients and induced birefringence are found using different sets of these coefficients. Misunderstandings related with the so-called photoelastic coefficients are resolved and new expressions for these coefficients are found. It is shown that the incorrect use of these coefficients for different pump beam distributions can lead to significant discrepancies for thermally induced birefringence. It is also shown that common use of the generalized thermo-optic coefficients significantly overestimates the values of optical power of thermal lenses when they are applied to the laser rods with lengths several times longer than their diameter.

Keywords: solid-state laser, thermal effect, photoelastic effect, plane strain, plane stress

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1. Introduction

The non-uniform temperature distribution in the active element (AE) of solid-state lasers (SSL) under the optical pumping causes undesirable thermal stresses, deformations and end-face bulging of the AE. Then the refractive index of AE is changing due to the joint action of linear thermo-optic (TO) and photoelastic (PE) effects. Different aspects of thermal effects in SSL are studied in many laser monographs and reviews [1–13]. The thermal lensing (TL) problem was first investigated for glass lasers [1, 5, 9, 10] when flash lamps were used for pumping of a long AE. Therefore, the plane strain (PSn) approximation for cylindrical rods [14] was used for the description of thermally induced stresses and deformations. In this approximation the initially plane faces of the AE remain plane for axisymmetrical pumping. In the description of the refractive index change with temperature the TO coefficient $\beta_\sigma \equiv (\partial n/\partial T)_{\sigma=0}$ is commonly referenced in the literature [1, 2], because it may be easily measured experimentally using the free expansion of the specimen at zero stress. The TO coefficient $\beta_\varepsilon \equiv (\partial n/\partial T)_{\varepsilon=0}$ at zero strain is used less often [5, 8]. Unfortunately, TO coefficients are used more often without the specification and are simply denoted as $\mathrm{d}n/\mathrm{d}T$ [6, 7, 9–10, 12, 13], $\partial n/\partial T$ [3, 4, 11] or $\delta n/\delta T$ [15].

Koechner (see [2] and references therein to the original papers) and Foster with Osterink [16] were the first ones who developed a theoretical model for the thermal lensing and birefringence in the Nd:YAG crystal grown in the crystallographic direction [111]. The mechanically isotropic properties of YAG allowed the use of the standard PSn approximation for the stress distributions with the general axisymmetrical temperature distribution. However, the investigations of these authors were restricted to the uniform pumping. It was shown that under the uniform thermal loading of the cylindrical AE with temperature independent thermal conductivity the quadratic radial dependence of the temperature distribution is achieved and that radial dependences of stresses and deformations are also quadratic for this case. The TO part of index change was described using the coefficient β_σ and the fact that Nd:YAG is a cubic crystal of symmetry class *m*3*m* was used in deriving the expressions for the PE part of index change. Nondimensional PE coefficients $C_{r,\theta}$ (in front of radially parabolic temperature terms) were introduced in [16] for the description of the PE part of the thermally induced refractive index change.

Afterwards they were accepted in the first edition of Koechner's monograph [2] and have been widely used till now, even in the graduate texts for students [17, 18]. It should be noted that in the practically unknown paper [11] both grown directions [001] and [111] were considered for rod and disk shapes of AE made from YAG and Al_2O_3. The results of [19] were partly reproduced later in the monograph [1].

Around that time the studies of the TL in heated windows of a high-power laser have started [20, 21]. The plane stress (PSs) approximation was used for the description of optical distortion in thin disk shape crystal windows neglecting their mechanical and photoelastic anisotropy. A new generalized parameter $\chi = \chi_1 + \chi_2 + \chi_3$ was introduced [22] for the description of the optical distortion effect in windows, where $\chi_1 = \mathrm{d}n/\mathrm{d}T$ is the temperature derivative of the refractive index at zero stress, $\chi_2 = (n_0-1)\,\alpha_T\,(1+\nu)$ is the thermal expansion (bulging) term in the PSs approximation, and χ_3 is the stress-optic term (n_0 is the initial refractive index, α_T is the linear expansion coefficient, and ν is the Poisson's ratio). This approach was further developed by Klein [23, 24]. As will be shown later, the expression for the polarization averaged PE part of the refractive index change obtained in [24] is not correct for the case of a long rod.

The generalized TO coefficients $\chi_{r,\theta}$ were proposed in [25]. These coefficients incorporated the TO coefficient $\mathrm{d}n/\mathrm{d}T$, the bulging term χ_2 and the PE part of the refractive index expressed through the $C_{r,\theta}$ with reference to [2]. Several inaccuracies were made in this original proposal (see a detailed discussion later). It was properly indicated in the review paper [8] that the "PE constants" $C_{r,\theta}$ are different when using PSn or PSs approximations. However, it was also claimed in this review that "W. Koechner published incorrect values of these coefficients in his reference book [2] because the temperature term in the Hook law has been omitted". In spite of the above remarks, the coefficients $C_{r,\theta}$ and $\chi_{r,\theta}$ were widely used in the previous form, see as examples [9–13, 18]. It may be due to the impression which has arisen from the widely referenced works of Cousins et al. [25, 26] that end-pumping of SSL requires the obligatory use of the PSs approximation with the generalized TO coefficients $\chi_{r,\theta}$, incorporating the bulging term χ_2 and the unspecified $\partial n/\partial T$ and $C_{r,\theta}$. The main goal of a recent paper [27] was to verify the existing analytical expressions due to Koechner and Foster & Osterink using finite-element simulations. The conclusion "that the Koechner and Foster & Osterink treatments are correct, and that Chenais et al. made mistakes in their derivation of the thermally-induced strain" was made in this paper. This confusion was resolved in papers [28, 29] though paper [27] was not known to us during the writing of these papers. It was shown that different, but in principle correct, expressions for the PE coefficients $C_{r,\theta}$ using the same PSn approximation were obtained due to the use of different TO coefficients (TOC) and different descriptions of the PE effect: $(\partial n/\partial T)_{\sigma=0}$ (and the piezo-optic variant of the PE effect) was used in [2, 16], just when $(\partial n/\partial T)_{\varepsilon=0}$ and the elasto-optic variant of the PE effect was used in [8].

In this paper, the correct use of linear thermo-optic and photoelastic effects for the description of thermally induced refractive index change is briefly discussed and the analytical relations between TOC at zero stresses and zero strains are found for all classes of cubic crystals. It is shown that the use of PE coefficients $C_{r,\theta}$ for different (not only for parabolic) temperature distributions is invalid and leads to significant discrepancies for thermally induced birefringence. The examples of inconsistent usage of the generalized TO coefficients $\chi_{r,\theta}$ are discussed. It is shown that the direct use of these coefficients significantly overestimates the values of optical power of the thermal lens when the PSs approximation is applied to the laser rods with lengths several times longer than their diameter.

2. Photoelastic effect and relations between thermo-optic coefficients

The constitutive equations of the linear theory of thermoelasticity for homogeneous crystals have very clear and short expressions if they are written in a tensor form. The Hooke's law [30, 31] was extended by Duhamel and Neumann to include the first order linear effect of thermal loading. This generalized Duhamel–Neumann law states that the total strain $\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ at the point of a solid consists of the stress-induced elastic strain $\varepsilon^{\sigma}_{ij} = s_{ijkl}\sigma_{kl}$ and the strain caused by the free thermal expansion $\varepsilon^{T}_{ij} = \alpha^{T}_{ij}(T-T_r)$: $\varepsilon_{ij} = \varepsilon^{\sigma}_{ij} + \varepsilon^{T}_{ij}$ [32]. Here u_i is the deformation displacements, s_{ijkl} is the components of the 4th-rank compliance tensor, σ_{kl} is the stress tensor and α^{T}_{ij} is the coefficients of linear thermal expansion, T_r is the temperature field at which the body is stress free and strain free. The inverted form of the Duhamel–Neumann law is given in an indicial form by $\sigma_{ij} = c_{ijkl}\varepsilon^{\sigma}_{ij}$, where c_{ijkl} is the components of the 4th-rank stiffness tensor [32]. The summation convention where repeated indexes indicate summation is implied throughout this paper.

The change of the relative dielectric impermeability tensor B_{ij} is commonly used for the phenomenological description of the optical effects induced in crystals [30]. Unfortunately, the contributions of

TO and PE effects to the thermally induced change ΔB_{ij} are generally treated separately as independent actions (see, for example, recent papers [27, 33, 34]). The change of the refractive index due to the TO effect for the temperature change $\Delta T = T - T_r$ is calculated first [27, 33, 34] as

$$\Delta n^{(\mathrm{TO})} = (\mathrm{d}n/\mathrm{d}T)\Delta T \quad (1a)$$

without specifying dn/dT. Then, it is assumed in [27, 33, 34] that the components of the optical indicatrix can be calculated using the "equivalent" expressions

$$\Delta B_{ij} = \pi_{ijlk}\sigma_{kl} \text{ or } \Delta B_{ij} = p_{ijkl}\varepsilon_{kl}, \quad (1b)$$

where π_{ijlk} and p_{ijkl} are the components of the piezo-optic and of the elasto-optic tensors with the following comments: "Normally one chooses one or the other formulation for use in a particular problem but, if done properly, the results will be identical" [27].

As explained in [29], if one wants to use stress or total strains in the description of the thermally induced photoelastic effect, the following expressions should be used:

$$B_{ij}(T,\sigma_{kl}) = B_{ij}^{(0)} + (\partial B_{ij}/\partial T)_{Tr,\sigma=0}\,\Delta T + (\partial B_{ij}/\partial \sigma_{kl})_{Tr,\sigma=0}\,\sigma_{kl} + \ldots, \quad (2a)$$

$$B_{ij}(T,\varepsilon_{kl}) = B_{ij}^{(0)} + (\partial B_{ij}/\partial T)_{Tr,\varepsilon=0}\,\Delta T + (\partial B_{ij}/\partial \varepsilon_{kl})_{Tr,\varepsilon=0}\,\varepsilon_{kl} + \ldots\,. \quad (2b)$$

Here $B_{ij}^{(0)}$ is an initial impermeability tensor of the crystal at the reference temperature T_r and zero stresses and strains, the second (different, in principle) terms describe the change due to the linear TO effect and the last terms (generally, also different) describe the change due to the linear piezo-optic or elasto-optics effects [31]. Thus, the final results will be identical only if these differences will be carefully taken into account.

In general, the piezo-optic $\pi_{ijkl} = (\partial B_{ij}/\partial \sigma_{kl})_{Tr,\sigma=0}$ and the elasto-optic $p_{ijkl} = (\partial B_{ij}/\partial \varepsilon_{kl})_{Tr,\varepsilon=0}$ tensors are symmetric with respect to i and j, but not necessarily symmetric with respect to k and l [35], due to the contribution of rotation of the volume element in an optically anisotropic medium. This symmetry is valid for cubic crystals only. A cubic crystal with an initial refractive index n_0 does not change its symmetry during free expansion ($\sigma_{kl} = 0$) and remains optically isotropic. Therefore, the change of an impermeability tensor with temperature change due to the linear TO effect can be expressed in this case as

$$B_{ij}^{(T,\,\sigma=0)} = \delta_{ij} \,/\, [n_0 + \beta_\sigma \Delta T]^2, \quad (3a)$$

and the total impermeability tensor can be presented as

$$B_{ij}(T,\sigma_{kl}) = B_{ij}^{(T,\,\sigma=0)} + \pi_{ijkl}\sigma_{kl}. \quad (3b)$$

If the full strain tensor ε_{ij} is used [8], then an elasto-optic form of the impermeability tensor change should be used [29]:

$$B_{ij}(T,\varepsilon_{kl}) = B_{ij}^{(T,\,\varepsilon=0)} + p_{ijkl}\varepsilon_{kl}. \quad (3c)$$

Here

$$B_{ij}^{(T,\,\varepsilon=0)} = \delta_{ij}/[n_0 + \beta_\varepsilon \Delta T]^2. \quad (3d)$$

Finishing the discussion of backgrounds of TO and PE effects, it is now appropriate, first, to note that it is mistakenly assumed in papers [33, 34] that the total strains ε_{kl} are expressed as $\varepsilon_{kl} = (\partial u_k/\partial x_l + \partial u_l/\partial x_k)/2 + \varepsilon_{kl}^T$ (Eq. (2) in [33, 34]) with the redundant free expansion term. Therefore, the constitutive Eqs. (3) and (4) in [33] are only valid if this additional term is switched off. Second, it is especially strange that the authors of [27], despite explicitly indicating difference between elastic ε_{kl}^σ and total ε_{kl} strains, do not recognize that Eq. (1b) will be actually equivalent if ε_{kl} were changed to ε_{kl}^σ [29].

The cubic sesquioxide Sc_2O_3, Lu_2O_3, and Y_2O_3 crystals and their ceramics were used in the past decade as host materials of SSL. These media have thermal conductivity comparable with that of YAG, whereas a Yb-doped gain bandwidth is significantly larger. This feature allows obtaining pulses with duration down to 50 fs at the oscillator output in the mode-locking regime. Thermally induced depolarization and thermo-optic properties of sesquioxide class m3 single crystals and ceramics are studied in recent papers [36–38]. Therefore, finding relations between β_σ and β_ε is very relevant for such sesquioxide class m3 single crystals.

It should be noted that the elasto-optic and piezo-optic matrices of m3 and class 23 crystals are not symmetric and have four independent components. Using the method proposed in [29] and Eq. (3), the required relation

$$\beta_\sigma = \beta_\varepsilon - \alpha_T n_0^3 (p_{11} + p_{12} + p_{13})/2 \quad (4a)$$

is easily found. Using the relation $p_{mn} = \pi_{mk}c_{kn}$ [30], Eq. (4a) may be written as

$$\beta_\sigma = \beta_\varepsilon - (\alpha_T n_0^3/2)\,(\pi_{11} + \pi_{12} + \pi_{13})\,(c_{11} + 2c_{12}), \quad (4b)$$

where c_{ij} is the components of the stiffness tensor.

The previous result [29] follows easily from Eq. (4) if the equality $p_{13} = p_{12}$ is taken into account [30]. It is also seen that Eq. (4) does not depend on p_{44} or π_{44}. It is due to the fact that only principal strains and stresses exist during free expansion or hydrostatic pressure of a cubic crystal. Unfortunately, we could not find in the literature any data concerning the values of elasto-optic or piezo-optic components of such crystals.

TO and expansion coefficients of YAG were measured many times and their values are widely ranged: $\beta_\sigma = (7.3–12.1) \times 10^{-6}$ K^{-1} and $\alpha_T = (5.8–9.9) \times 10^{-6}$ K^{-1} at 300 K. Meanwhile, paper [39] is the only one referenced in the literature [1–13, 15–19] for the values of measured elasto-optic coefficients (EOC): $p_{11} = -0.029$, $p_{12} = 0.0091$, $p_{44} = -0.0615$. However, different values $p_{11} = 0.060$ and $p_{12} = 0.022$ were measured in [40]. To the best of our knowledge, there are no papers in which the results of [40] were discussed or refuted. Moreover, it was noted in the later work [41] that "Introduction of the ions Er^{3+}, Nd^{3+}, Cr^{3+}, Sc^{3+} into the garnet hosts significantly increases the PE interaction parameter in comparison with the original crystals." The following values were given for the Er:YAG crystal: $p^*_{11} = -0.081$, $p^*_{12} = -0.035$, $p^*_{44} = -0.082$. The values of elasto-optic parameters for other garnets from this paper are referenced in [9], but for the YAG crystal the previous data from [39] is presented. It is seen that the values of EOC measured in these papers differ significantly not only in their magnitude but also in signs. We do not think that the doping can change so strongly the values of elasto-optic parameters, but if so, then new measurements of EOC for YAG crystals and first measurements for sesquioxide single crystals with different doping ion concentrations are needed. In a recent paper [42] the following values for polycrystalline YAG are obtained: $p^{pc}_{11} = -0.0627$, $p^{pc}_{12} = 0.0260$, $p^{pc}_{44} = -0.0444$. These values do not fulfill the equality $p_{44} = (p_{11}-p_{12})/2$ for isotropic solids [30]. We think that this situation may be possible for polycrystalline aggregates, but it is hardly probable for polycrystals the grains of which are much smaller than the wavelength at which p_{ij} is determined [43].

Thus, the difference $\beta_\alpha = \beta_\varepsilon - \beta_\sigma = \alpha_T n_0^3(p_{11} + 2p_{12})/2$ changes not only the value but also its sign when different values of p_{ij} from [39–41] are used. The values of the ratio $|\beta_\alpha|/\beta_\sigma$ are equal to 0.04, 0.4, and 0.6 if data from [39–41] and the highest (for α_T) and lowest (for β_σ) values are used in calculations. However, the microscopic relation proposed in [8] between TOC for YAG gives a very high value of $\beta_\varepsilon = 31.5 \times 10^{-6}$ K^{-1} as compared with the value of $\beta_\sigma = 9 \times 10^{-6}$ K^{-1} used in that paper.

3. Plane strain and plane stress approximations in thermal lensing

The PSn and PSs approximations of linear thermoelasticity are widely used for the description of thermal refractive index changes in an elastically isotropic AE of SSL under axially symmetric pumping. In the polar coordinate system the strain and stress tensors for both approximations have only the diagonal components and are expressed [14] through the local temperature $\widetilde{T}(r) = T(r) - T_r$ and average temperature $T^{(r)}$ changes in the circles with the radius r and R, the rod radius. PSn and PSs approximations are strictly valid for the temperature distributions which do not change along the axial direction. The expressions for stress tensors [14] can be written in a particularly simple symmetrical form [29] if the following definitions are introduced:

$$T^{(r)} = \frac{2}{r^2}\int_0^r \widetilde{T}(r) r \mathrm{d}r, \tag{5a}$$

$$\breve{T}(r) = T^{(r)} - \widetilde{T}(r) = -\frac{1}{r^2}\int_0^r \frac{\mathrm{d}\widetilde{T}(r)}{\mathrm{d}r} r^2 \mathrm{d}r. \tag{5b}$$

Then, simple expressions for the refractive index in the plane strain (superscript $j = 1$) or plane stress (superscript $j = 2$) approximations may be found [29]:

$$n^{(j)}_{r,\theta} = n_0 - A_1^{(\sigma j)} T^{(R)} + [\beta_\sigma + A_1^{(\sigma j)}]\widetilde{T} \pm A_2^{(\sigma j)} \breve{T}. \tag{6}$$

Here upper (+) and lower (–) signs describe the radial and tangential components of index change and

$$A_1^{(\sigma 1)} = \frac{n_0^3 \alpha_T E}{12(1-\nu)}(4\pi_{11} + 8\pi_{12} - \pi_{44}), \tag{7a}$$

$$A_2^{(\sigma 1)} = \frac{n_0^3 \alpha_T E}{12(1-\nu)}(\pi_{11} - \pi_{12} + 2\pi_{44}), \tag{7b}$$

$$A_1^{(\sigma 2)} = \frac{n_0^3 \alpha_T E}{12}(2\pi_{11} + 4\pi_{12} + \pi_{44}), \tag{7c}$$

$$A_2^{(\sigma 2)} = \frac{n_0^3 \alpha_T E}{12}(\pi_{11} - \pi_{12} + 2\pi_{44}), \tag{7d}$$

where E is the Young's modulus.

The above coefficients $A_1^{(\sigma 1,2)}$ may be easily expressed through the elasto-optic coefficients [29]. A superscript σ in these coefficients means that they should be used together with the TO coefficient β_σ in Eq. (6). If Eqs. 3(c, d) with β_ε are used in the derivation of expressions for the refractive index, then for these coefficients $A_1^{(\varepsilon 1,2)}$ a superscript ε is used [29].

As shown in [29], the equality $\beta_\varepsilon + A_1^{(\varepsilon j)} = \beta_\sigma + A_1^{(\sigma j)}$ is valid.

It is obvious that ceramic lasers [44] will be very widely used in the near future. Therefore, a short review of the formulae used previously for the description of the PE effect in isotropic solid media is appropriate. Using Eqs. (6) and (7) it is easy to get explicit expressions for the constants $B_\parallel$ and $B_\perp$ used in [20, 21]:

$$B_\parallel \equiv -(\partial n/\partial \sigma_\parallel)_{\tilde{T}=0} = \frac{n_0^3}{2}\pi_{11},$$
$$B_\perp \equiv -(\partial n/\partial \sigma_\perp)_{\tilde{T}=0} = \frac{n_0^3}{2}\pi_{12}. \quad (8a)$$

Using the relations between π_{ij} and p_{ij} for isotropic solids [29], these constants can be expressed through the EOC:

$$B_\parallel \equiv (n_0^3/2E)\,(p_{11} - 2\nu p_{12}),$$
$$B_\perp \equiv (n_0^3/2E)\,[(1-\nu)\,p_{12} - \nu p_{11}]. \quad (8b)$$

In Russian literature the constants $C_1 = -B_\parallel$ and $C_2 = -B_\perp$ with the opposite sign were used more often. So, the widely used [1, 3, 4] constants

$$W = \beta_\sigma + (n_0 - 1)\alpha_T, \quad (9a)$$

$$P \equiv \beta_\sigma - \frac{\alpha_T E}{2(1-\nu)}(C_1 + 3C_2), \quad (9b)$$

$$Q = \frac{\alpha_T E}{2(1-\nu)}(C_1 - C_2) \quad (9c)$$

are simply expressed as

$$P \equiv \beta_\sigma + A_1^{(\sigma 1)},\ Q = -A_2^{(\sigma 1)}. \quad (9d)$$

Then the thermally induced refractive index for isotropic solids in the plane strain approximation may be presented as

$$n_{r,\theta}^{(1)}(r) = n_0 + \beta_\sigma T^{(R)}$$
$$+ P(\tilde{T} - T^{(R)}) \pm Q(\tilde{T} - T^{(r)}). \quad (10a)$$

Taking into account that the longitudinal component of the strain tensor in this approximation is $\varepsilon_{zz}^{(1)} = \alpha_T T^{(R)}$ [45], the local change of the optical path (without taking into account the end-face-bulging) may be expressed as [1, 3, 4, 46]

$$\mathrm{OP}_{r,\theta}^{(1)}(r) = [(W-P)T^{(R)} + P\tilde{T} \pm Q(\tilde{T} - T^{(r)})]L. \quad (10b)$$

The authors of [46] incorrectly assumed that these expressions can be used for mechanically anisotropic cubic crystals (LiF, KCl, CaF_2) if their anisotropic PE properties are taken into account. The details of analytical simulations of the TO characteristics of the cylindrical and disk AE are absent in [5, 46]. Therefore, the validity of the presented expressions was checked by comparing the formulae in [5, 46] with our expressions (6) and (7). It can be shown that for the rod type AE $Q^{(1)} \equiv A_2^{(\sigma 1)}$ and the TO coefficient $\beta \equiv \beta_\sigma$ should be changed to β_ε in the expressions for W and $P^{(1)}$. It should be also noted that the bulging term $\chi_{\mathrm{bg}}^{(2)} = (n_0 - 1)(1+\nu)\alpha_T$ in the PSs approximation is introduced into $P^{(2)}$ [5, 46].

In a long series of papers (see, for example, [23, 24]) Klein promoted the idea that the (111) plane for all cubic crystals has isotropic elastic and PE properties. Therefore, it was assumed (by analogy with Eq. (8)) that optical path distortion may be described by introducing two new piezo-optic coefficients $\pi_\parallel \equiv (\pi_{11} + \pi_{12} + \pi_{44})/2$ and $\pi_\perp \equiv (\pi_{11} + 5\pi_{12} - \pi_{44})/6$ for stresses applied parallel and perpendicular to the polarization axis, respectively. Then, thermal lensing coefficients $\chi_\pm$ for "thick" windows

$$\chi_+^{(1)} = \beta_\sigma + \frac{n_0^3 \alpha_T E}{4(1-\nu)}(\pi_\parallel + 3\pi_\perp), \quad (11a)$$

$$\chi_-^{(1)} = \frac{n_0^3 \alpha_T E}{4(1-\nu)}(\pi_\parallel - \pi_\perp) \quad (11b)$$

and for "thin" windows

$$\chi_+^{(2)} = \beta_\sigma + \chi_{\mathrm{bg}}^{(2)} + (n_0^3 \alpha_T E/4)(\pi_\parallel + \pi_\perp), \quad (12a)$$

$$\chi_-^{(2)} = (n_0^3 \alpha_T E/4)(\pi_\parallel - \pi_\perp) \quad (12b)$$

were introduced. The coefficients $\chi_+^{(1,2)}$ combine TO coeffients (TOC), bulging terms ($\chi_{\mathrm{bg}}^{(1)} = 0$) plus the average PE effect for two polarizations. The coefficients $\chi_-^{(1,2)}$ characterize the stress-induced birefringence. Comparing with our expressions it is easy to see that $\chi_-^{(1,2)} = A_2^{(\sigma 1,2)}$ and $\chi_+^{(2)} = A_1^{(\sigma 2)}$. Unfortunately, the second term in $\chi_+^{(1)}$ does not equal $A_1^{(\sigma 1)}$. The reason of this mistake is the impossibility to present, in general, the change of the impermeability tensor for cubic crystals in the same form as for isotropic solids when $\pi_{44} = \pi_{11} - \pi_{12}$ and $\pi_\parallel = \pi_{11}$, $\pi_\perp = \pi_{12}$. Therefore, Eq. (11) can only be used for isotropic solids, as in [47] for glasses.

It should be noted that the general expressions for thermal stresses and strains in the plane strain approximation for hollow and bulk rods were described in detail in the little-known monograph [1] and paper [19]. The piezo-optic version (2a) of the PE effect and piezo-optical coefficients were used consistently in those works. Therefore, relations between β_σ and β_ε were not needed for the accurate description of thermal optical distortions in YAG crystals. It has also been understood for a long time that the use of the standard PSn approximation for isotropic solids for the description of thermoelastic stresses in mechanically anisotropic crystals is incorrect and that search of new solutions is required [48]. This hard task was solved in the PSn approximation for cubic [49] and generally anisotropic [50] crystals for parabolic temperature distribution only. The last solution was recently used in the series of papers (see, for example, [51]).

4. Refractive index change for special pump beam distributions

In the previous section the PSn and PSs approximations for general axisymmetrical temperature distribution were analyzed. However, the parabolic temperature distribution has been mainly used [2, 16]. For this case the PE coefficients $C_{r,\theta}$ were introduced [16] which were later attacked in review [8].

A particularly simple solution of the heat transfer equation can be found [28] for the polynomial radial heating distribution where the normalized thermal loading on the rod axis is expressed through the full loading power P_h of a cylindrical AE with radius R and length L. The known results for uniform and parabolic pump beam distributions follow from this solution. The solutions for top-hat and Gaussian pump beam distributions can also be found [52].

The radial temperature distributions for these four pump distributions with the same P_h = 60 W for a cylindrical YAG rod with R = 2 and L = 10 mm are presented in Fig. 1. The following parameters were used in numerical simulations: thermal conductivity coefficient k_0 = 0.105 W/(cmK), coefficient of the Newton's law of heat transfer h = 2 W/(cm^{2}K), the radii of top-hat and Gaussian pump beams $r_p = w_p$ = 1 mm. It is seen that the temperature at the rod edge does not depend on the pump beam distribution inside the rod; it depends on the full thermal load P_h and the coefficient h only. It is also obvious that transverse temperature distribution is parabolic for a uniform pump only. The radial temperature gradients are very different. Therefore, the thermally induced stresses, strains and refractive index changes are also very different.

The refractive index distributions for radial and tangential directions are presented in Fig. 2. They were calculated using Eq. (6), EOC from [2, 8, 39] and the values β_σ = 8.4 ppm/K and α_T = 6.4 ppm/K at T_r = 300 K

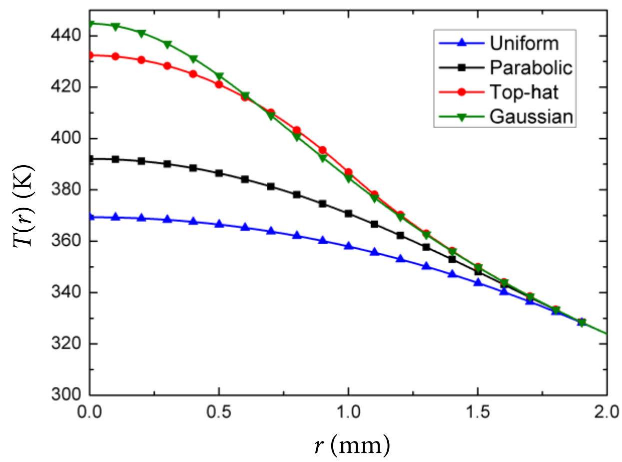

Fig. 1. Radial temperature distribution for uniform (up-pointing triangle, blue online), parabolic (square), top-hat (circle, red online) and Gaussian (down-pointing triangle, green online) thermal loading with the same total power.

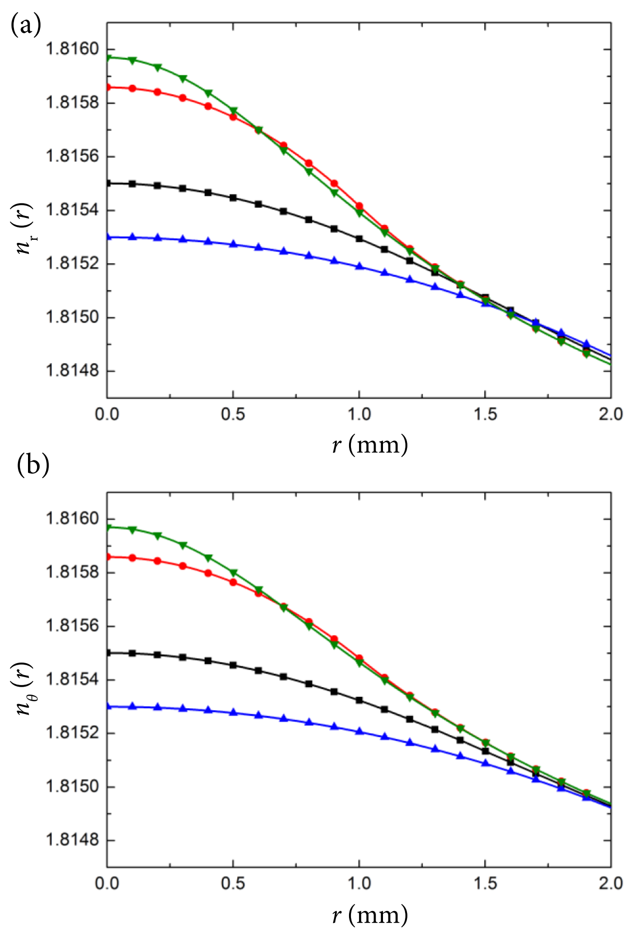

Fig. 2. Radial distribution of the radial (a) and the tangential (b) refractive index for uniform (up-pointing triangle, blue online), parabolic (square), top-hat (circle, red online) and Gaussian (down-pointing triangle, green online) thermal loading with the same total power.

from [53]. It is seen that both induced indices are higher than the initial index $n_0 = 1.8147$ used in the numerical simulations. Therefore, the standard presentation of thermal index variation as $n(r) = n_0(1-\gamma r^2)$ (see, for example, [2, 16–18]) is not valid even for parabolic temperature distribution. It is necessary that at the optical axis the stresses and strains in radial and tangential directions would be equal as $n_r(0) = n_\theta(0)$. Sometimes, in the literature numerically obtained results do not satisfy these requirements for an axially symmetric case. It is also seen that, in general, index variation is not proportional to radial temperature variation as it is often assumed [2, 8] where the local change of temperature $\Delta T(r) = T(r)-T_r$ or the refractive index $\Delta n^{(1,2)}_{r,\theta}(r) = n^{(1,2)}_{r,\theta}(r) - n_0$ is not clearly distinguished from the nonlocal temperature $\delta T(r) = T(0)-T(r)$ or the index difference

$$\begin{aligned}\delta n^{(1,2)}_{r,\theta}(r) &= n^{(1,2)}_{r,\theta}(0) - n^{(1,2)}_{r,\theta}(r)\\ &= [\beta_\sigma + A_1^{(\sigma 1,2)}]\delta T(r) \mp A_2^{(\sigma 1,2)}\breve{T}(r).\end{aligned} \tag{13}$$

It is also clear that the generalization of the simplified formula

$$\begin{aligned}\delta n^{(1,2)}_{r,\theta\, C}(r) = (\beta_\sigma \\ + 2n_0^3\alpha_T C^{(\sigma 1,2)}_{r,\theta})\delta T(r),\end{aligned} \tag{14}$$

which is valid for parabolic temperature distribution to the general temperature distribution, is incorrect in the general case, $C^{(\sigma 1,2)}_{r,\theta}$ is the so-called photoelastic coefficients (PEC) in which a superscript (1) designates the use of the PSn approximation, (2) is the use of the PSs approximation and a superscript σ means that this PEC should be used together with β_σ. If the β_ε is used in calculations, then another set of PE coefficients $C^{(\varepsilon 1,2)}_{r,\theta}$ should be used in Eq. (14).

The difference between consistent and simplified approaches can be seen more clearly if instead of looking at Eqs. (13) and (14) overwhelmed by a high enough TO coefficient β_σ, expressions $\delta n^{(-)}_{r,\theta}(r)$ and $\delta n^{(-)}_{r,\theta\,C}(r)$ without it are used in the PSn approximation (Fig. 3). It follows from these results that the use of a simplified formula for the Gaussian pump beam leads to significant differences with the results obtained when the PSn approximation is used consistently. The positive temperature difference $\delta T(r) > 0$ increases monotonically with r. Therefore, $\delta n^{(-)}_{rC}(r)$ and $\delta n^{(-)*}_{rC}(r)$ increase monotonically with r (Fig. 3(a), (b)) because $C_r^{(\sigma 1)}(r) = 0.0176$ for standard values of EO coefficients (EOC) [39] and $C_r^{(\sigma 1)*}(r) = 0.0086$ for an alternative set of p^*_{ij} [41]. Similarly, the tangential components of $\delta n^{(-)}_{\theta C}(r)$ and $\delta n^{(-)*}_{\theta C}(r)$ decrease monotonically with r due to the negative values of PE coefficients $C_\theta^{(\sigma 1)} = -0.0025$ and $C_\theta^{(\sigma 1)*} = -0.0179$. At the same time, the variation of $\delta n^{(-)}_\theta(r)$ is non-monotonical with r, even the sign of its value is changed. This behaviour is caused by the nonlocal nature of $\breve{T}(r)$ which depends integrally on the temperature gradient.

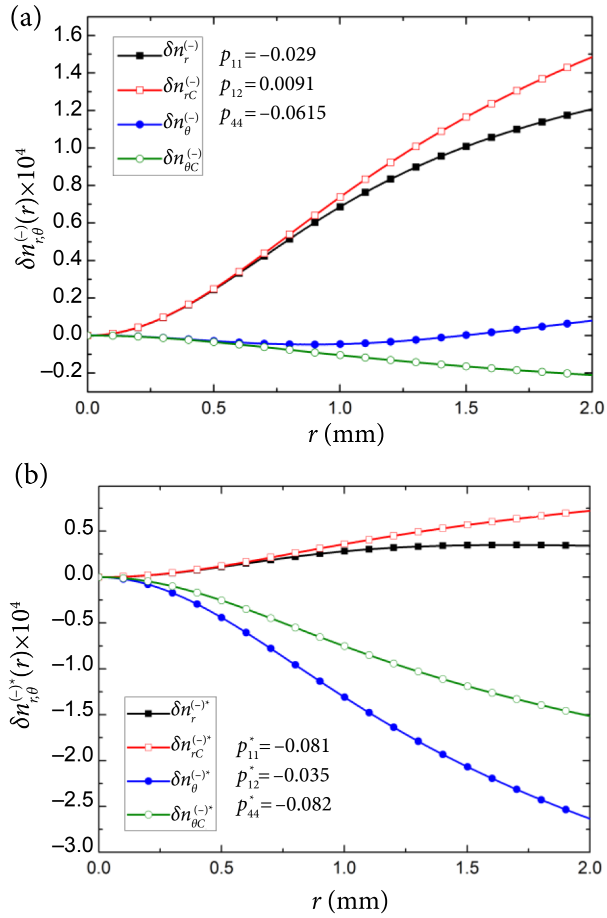

Fig. 3. Radial distribution of the radial (square) and the tangential (circle) refractive index difference due to the photoelastic effect calculated from the simplified formula (hollow) and the full formula (solid) using (a) standard elasto-optic coefficients and (b) alternative ones.

The difference between consistent and simplified calculation of the induced thermal index change affects mainly the value and radial behaviour of the induced birefringence $\delta n(r) = n_r^{(1)}(r) - n_\theta^{(1)}(r)$ (Fig. 4). It is seen that for the standard set of p_{ij} the simplified calculation of the birefringence

$$\begin{aligned}\delta n_C(r) &= 2n_0^3\alpha_T(C_r^{(\sigma 1)} - C_\theta^{(\sigma 1)})\delta T(r)\\ &= -4n_0^3\alpha_T C_B^{(1)}\delta T(r)\end{aligned} \tag{15}$$

gives higher values as compared with the consistent use of Eq. (6); here $C_B^{(1)} = (C_\theta^{(\sigma 1)}-C_r^{(\sigma 1)})/2$. The situation changes inversely when the alternative set of p^*_{ij} is used.

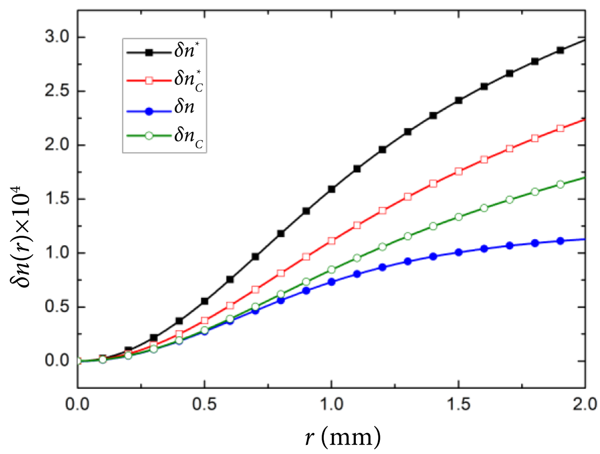

Fig. 4. Radial distribution of induced birefringence calculated using the simplified formula (hollow) and the full formula (solid) with standard (circle) elasto-optic coefficients and an alternative set (square).

It is seen that using different sets of EOC and different approaches to the calculation for the same Gaussian pump the obtained values of induced birefringence in the YAG rod can differ significantly (~3 times) at the rod edge (Fig. 4). It should also be mentioned that due to Eq. (4) the $C_B^{(1)} = (C_\theta^{(\sigma,\varepsilon 1)} - C_r^{(\sigma,\varepsilon 1)})/2$ [29] is not "coincidentally the same formula obtained by Koechner from incorrect expressions" [8, p. 146].

5. On the use of generalized TO coefficients

As mentioned in the Introduction, the generalized TO coefficients $\chi_{r,\theta}$ were first proposed in [25] for top-hat pumping of AE: "The optical path difference is given by OPD = $\Delta T L \chi$ (2*) where ΔT is the temperature difference from center to edge, and χ is the TO coefficient. χ contain terms for refractive index changes, axial expansion, and stress-induced changes in refractive index $\chi = \mathrm{d}n/\mathrm{d}T + (n-1)(1+\nu)\alpha_T + n^3\alpha_T C_{r,\theta}$ (3*). Here dn/dT is the thermal coefficient of the index of refraction, ν is Poisson's ratio, α_T is the coefficient of thermal expansion, and $C_{r,\theta}$ is the photoelastic coefficient [2]."

Several inaccuracies were made in this original proposal. First of all, two coefficients $\chi_{r,\theta}$ (not one χ) need to be defined. Second, the OPD and ΔT are not clearly defined. Therefore, the definition of the optical path OP(r) should be clearly stated. Let z_1 and z_2 be reference planes on the opposite sides of the laser rod [26]. Then, the optical path is defined as the integral $\mathrm{OP}(r, z_2 - z_1) = \int_{z_1}^{z_2} n(r,z)\mathrm{d}z$ at the constant r from plane z_1 to plane z_2 including intervals in the air and in the thermally excited AE. There are several possibilities to define the optical path difference. We will use in this paper the definition $\mathrm{OPD}(r) \equiv \mathrm{OPD}^{(+)}(r) = \mathrm{OP}(0) - \mathrm{OP}(r)$. Often the opposite definition $\mathrm{OPD}^{(-)}(r) = -\mathrm{OPD}^{(+)}(r)$ is used. It seems that in paper [25] the temperature difference and optical path difference were defined as $\Delta T \equiv \delta^* T = T(r) - T(R)$ and $\mathrm{OPD} \equiv \mathrm{OPD}^*(r) = \mathrm{OP}(r) - \mathrm{OP}(R)$, respectively, that is $\mathrm{OPD}^*(r) = -\mathrm{OPD}(r) + \mathrm{OP}(0) - \mathrm{OP}(R)$. Further, the coefficient dn/dT was not specified, the expression for the bulging term was obtained using the PSs approximation which is valid for very thin disks only, but the stress induced PE term with the Koechner's PE coefficients $C_{r,\theta}$ valid for long rods (and missing factor of 2 before them) was used. Besides, as follows from the previous discussion, the generalized TOC can be introduced for only parabolic temperature distributions. For the top-hat pump temperature distribution differs significantly from that of the parabolic one (Fig. 1). Thus, it was assumed in [25] that the TO coefficient χ may be used for different temperature distributions.

The definition of $\chi_{r,\theta}$ was updated in the review paper ([8], p. 115): "We would like to point out two important clarifications... Only the *plane strain* case was considered by Koechner. However, we saw that the plane stress case is closer to reality in end-pumped rods. Here we denote C_r and C_θ as the photoelastic constants valid for long and thin rods (the "Koechner case", that is when the plane strain approximation is valid), and C'_r and C'_θ as the photoelastic constants derived within the framework of the plane stress approximation. Since we are only interested in end pumping, we only consider the $C'_{r,\theta}$ constants in the following".

Thus, the expression $\mathrm{OPD}_{r,\theta}(r) = \chi_{r,\theta}^{(C)} L \delta T(r)$, where $\chi_{r,\theta}^{(C)} = \beta_\varepsilon + \chi_{\mathrm{bg}}^{(2)} + 2n_0^3\alpha_T C'_{r,\theta}$, was recommended by Chénais et al. for the use in the end pumped case in spite of the AE length ([8], p. 119). Taking into account the relation between β_σ and β_ε, the equality $\chi_{r,\theta}^{(C)} \equiv \chi_{r,\theta}^{(2)} = \beta_\sigma + \chi_{\mathrm{bg}}^{(2)} + \chi_{\mathrm{pe}}^{(2\sigma)}$ follows from this recommendation, where $\chi_{\mathrm{pe}}^{(2\sigma)} = 2n_0^3\alpha_T C_{r,\theta}^{(\sigma 2)}$ is the PE part. Thus, the optical path differences in the PSn (superscript 1) and PSs (superscript 2) approximations are given by

$$\mathrm{OPD}_{r,\theta}^{(1,2)}(r) = \chi_{r,\theta}^{(1,2)} L \delta T(r), \qquad (16)$$

where the generalized TOC are introduced: $\chi_{r,\theta}^{(1,2)} = \beta_\sigma + C_{\mathrm{bg}}\chi_{\mathrm{bg}}^{(1,2)} + \chi_{\mathrm{pe}}^{(1,2)}$. Using Eq. (4) and the definition $C'_{r,\theta} = C_{r,\theta}^{(\varepsilon 2)}$, it is easy to get the new expressions

$$C_r^{(\sigma 2)} = \frac{p_{11}(1-3\nu) + p_{12}(3-5\nu)}{16}, \qquad (17a)$$

$$C_\theta^{(\sigma 2)} = \frac{(5-7\nu)p_{11} + (7-17\nu)p_{12} + 8(1+\nu)p_{44}}{48}. \qquad (17b)$$

Now, using two sets of EOC [39, 41], the following values of PEC are obtained: $C_r^{(\sigma 2)} = 0.0033$, $C_\theta^{(\sigma 2)} = -0.0116$ and $C_r^{(\sigma 2)*} = 0.0329$, $C_\theta^{(\sigma 2)*} = 0.0133$. It is easy to show that $C_B^{(2)} = (C_\theta^{(\sigma,\varepsilon 2)} - C_r^{(\sigma,\varepsilon 2)})/2 = (1-\nu)C_B^{(1)}$, that is the birefringence parameter in the PSs approximation, is 26% lower than that in the PSn approximation. It also follows that for the alternative set of EOC the induced birefringence is 30% higher ($C_B^{(1,2)*}/C_B^{(1,2)} = 1.32$) keeping other parameters the same.

The value of the bulging coefficient C_{bg} in Eq. (16) depends on the approximation used: $C_{bg} = C^{(1)} = 0$ in the PSn and $C_{bg} = C^{(2)} = 1.0$ in the PSs approximations; $C_{bg} = C^{(P)} = 1/(1+\nu)$ if the free longitudinal expansion of the whole rod is assumed [6]; $C_{bg} = C^{(K)} = 2R/[L(1+\nu)]$ if the Koechner assumption is used [2]; $C_{bg} = C^{(H)} = 1/(1-\nu)$ if the term $\nabla\times\nabla\times\vec{u}$ in a steady state equilibrium equation is neglected ($\vec{u}$ is a displacement vector) [54].

To evaluate more precisely the contribution of the bulging term direct calculations of face bulging were performed using *COMSOL* Multiphysics for parabolic temperature distribution and different rod lengths with the rod radius $R = 2$ mm (Fig. 5(a)). The use of *COMSOL* Multiphysics software was validated [55] by reproducing the known numerical results for the so-called cubic cylinder with $L = 2R$ for the case of parabolic temperature distribution (see [56], pp. 223–239).

The face bulging $w(r)$ was normalized to the maximum sag $w^{(2)}(0) = (1 + \nu)\alpha_T\,|T_2|/2$ which is predicted by the PSs approximation [45]. The calculations showed that the bulging is close to the PSs prediction at $L/R < 0.5$ and saturates if $L > 2R$. Therefore, the ratio $C^{(B)} = w(0)/w^{(2)}(0)$ is close to 1 for a thin disk only if $L/R < 0.5$ (Fig. 5(b)). Thus, the bulging coefficient $C_{bg} = C^{(B)}$ obtained by direct numerical calculations diminishes very quickly with increasing the rod length, even faster than was proposed by Koechner. It is also seen that the value of $C_{bg} = C^{(H)}$ proposed in [54] is far from reality.

For the parabolic temperature distribution $\tilde{T}(r) = T_0 + T_2(r/R)^2$ the optical path difference is also parabolic $\mathrm{OPD}_{r,\theta}^{(1,2)}(r) = r^2/2f_{r,\theta}^{(1,2)}$, where $f_{r,\theta}^{(1,2)}$ is the focal lengths for radial and tangential polarizations. Thus, the optical power of a thermal lens is given by $D_{r,\theta}^{(1,2)} = 1/f_{r,\theta}^{(1,2)} = (-2LT_2/R^2)\chi_{r,\theta}^{(1,2)}$, that is proportional to the generalized TOC. In order to evaluate the value of optical power the numerical values of all other parameters of $\chi_{r,\theta}^{(1,2)}$ should be known.

The ratio of the focal lengths for tangential and radial polarizations is given by

$$\frac{f_\theta^{(1,2)}}{f_r^{(1,2)}} = \frac{1+\alpha_\beta(n_0-1)(1+\nu)C_{bg}+\alpha_\beta 2n_0^3 C_r^{(\sigma 1,2)}}{1+\alpha_\beta(n_0-1)(1+\nu)C_{bg}+\alpha_\beta 2n_0^3 C_\theta^{(\sigma 1,2)}}, \quad (18)$$

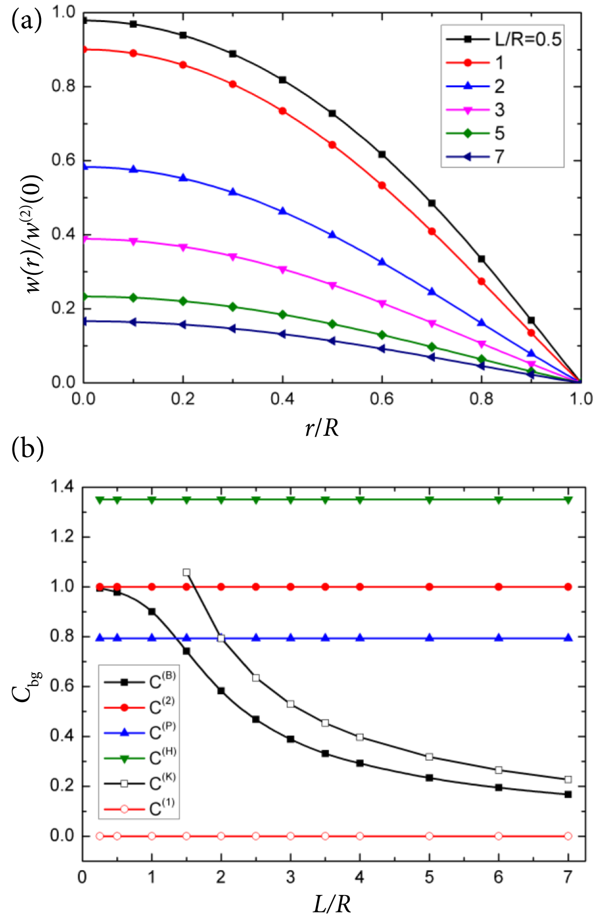

Fig. 5. Normalized bulging of the end faces for various lengths of the AE with the same radius $R = 2$ mm (a) and dependence of bulging coefficients on the length of the AE according to various approximations (b).

where the values of $\alpha_\beta = \alpha_T/\beta_\sigma$ lie in the range 0.48–1.36 as follows from the literature known to us. It is easy to see from Eq. (18) that this ratio is maximal for the PSn approximation when the bulging term is neglected ($C_{bg} = 0$) and it is minimal for the PSs ($C_{bg} = 1$). Obviously, for a rod shape AE the coefficients $C_{r,\theta}^{(\sigma 1)}$ should be used. The ratios (18) for AE with $R = 2$ and $L = 10$ mm ($C_{bg} = 0.23$) are presented in Fig. 6(a) and (b) when EOC from [39] or [41] were used, respectively. It is seen that these ratios are slightly higher for the alternative set of p_{ij}^*. It is seen (Fig. 6(a)) that for the standard set of p_{ij} this ratio does not exceed 1.34 in the whole range of α_β.

It follows from Eq. (18) that a significant error may be made when the PSs approximation is used for the calculation of values of the focal length of the TL instead of using a more correct formula with the correcting parameter $C_{bg}^{(B)}$ derived in this paper. It is seen from Fig. 6 that the focal length determined in the PSs approximation ($C_{bg}^{(2)} = 1.0$) may be twice or even more shorter than the length determined by using the appropriate

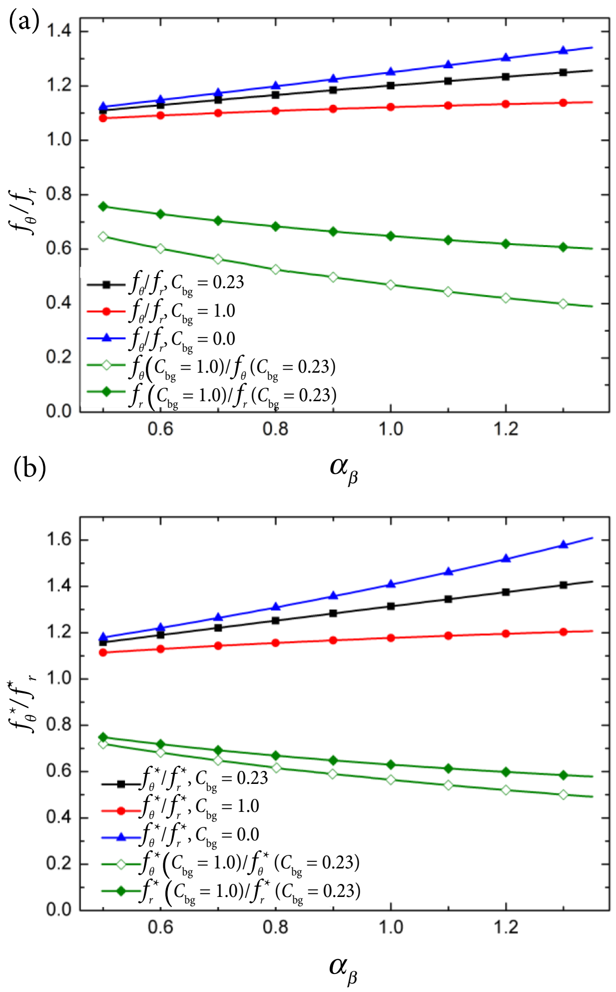

Fig. 6. Ratios of radial and tangential focal lengths for various approximations depending on the ratio of the thermal expansion coefficient to the thermo-optic coefficient at zero stress for (a) a set of standard elasto-optic coefficients and (b) an alternative one.

parameter $C_{bg}^{(B)}$. There are many papers in the literature (see, for example, [57]) where the above discussed incorrect approach is used. With reference to [8] the generalized TO coefficient $\chi_r^{(2)}$ is used for the Nd:YAG crystal with the radius R = 1.5 mm and the length L = 8 mm. For these rod parameters the coefficient $C_{bg}^{(B)} \approx 0.2$ is even lower than the one used in the numerical calculations (Fig. 6). The authors of [57] neglected the contribution of the PE term $\chi_{pe}^{(W)}$ = 1.8 ppm/K to $\chi_r^{(W)}$ = (7.3 + 8.0 + 0) though the value $C_r^{(W)}$ = 0.0195 was pointed out in the text. Then, instead of $(\partial n/\partial T)_\varepsilon = 31.5 \times 10^{-6}$ K^{-1} recommended in [8] they used (without any comments) dn/dT = 7.3 ppm/K and α_T = 7.5 ppm/K. Thus, the value of $\chi_r^{(W)}$ used in [57] is equal to 15.3 ppm/K instead of $\chi_r^{(C)}$ = (31.5 + 8.0 + 0.3) ppm/K that follows from the recommendation of [8] and $\chi_r^{(W*)}$ = (7.3 + 8.0 + 1.6) ppm/K if the PSs approximation is consistently used. The value $\chi_r^{(B)}$ = (7.3 + 0.2 × 8.0 + 1.6) ppm/K follows from our calculations with $C_{bg}^{(B)} \approx 0.2$ and $C_r^{(\sigma 1)}$ = 0.0176. Therefore, the optical power used in [57] is 1.5 times higher and would be even 3.8 times stronger than the one obtained in this paper if the recommendation of [8] were used.

6. Conclusions

Consistent application of the photoelastic effect and the plane strain or plane stress approximation for the description of a thermal change of the refractive index in the case of axisymmetric heat loading is analyzed in detail. The analytical relations between thermo-optic coefficients of the refractive index at zero stresses and zero strains are found for cubic crystals of all classes. These results may be interesting for researchers investigating thermal effects in very promising sesquioxide class m3 single laser crystals. It is shown that the ratio of the difference between thermo-optic coefficients at zero stress or zero strain to the thermo-optic coefficient at zero stress is significantly larger for alternative sets of YAG elasto-optic coefficients known in the literature as compared with a standard set of elasto-optic coefficients which is the only one used in the description of thermal lensing.

A detailed analysis of the analytical expressions for thermal radial and tangential changes of the refractive index in the [111] cut YAG crystal for general axisymmetric thermal loading is carried out. Misunderstandings related with the so-called photoelastic coefficients are eliminated. It is shown that the use of these coefficients for various pump beam distributions may lead to significant discrepancies for thermally induced birefringence as compared with the consistent use of plane strain or plane stress approximations.

The contribution of the bulging term into the generalized coefficient is analyzed numerically. It is noted that the usage of photoelastic and generalized thermo-optic coefficients is not as useful as it is widely assumed. It is also shown that the common use of generalized thermo-optic coefficients significantly overestimates the value of optical power of thermal lenses when the plane stress approximation is applied to the laser rods with lengths larger than their diameter.

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FOTOELASTINIO EFEKTO, PLOKŠČIŲ ĮTEMPIŲ IR DEFORMACIJŲ ARTINIŲ NAUDOJIMAS APRAŠYTI ŠILUMINĮ FOKUSAVIMĄ

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Santrauka

Naudojant tikslias išraiškas, aprašančias pjezo optinius ir elasto optinius efektus visų klasių kubiniuose kristaluose, surasti analitiniai ryšiai tarp lūžio rodiklio temperatūrinių išvestinių esant nuliniam įtempiui ir nulinei deformacijai. Šie ryšiai gali būti naudingi tirti šiluminius efektus perspektyviuose m3 klasės tipo lazerio kristaluose. Skaičiavimuose naudojami dažnai cituojami standartinis ir literatūroje rastas alternatyvus YAG elasto optinių koeficientų rinkiniai. Parodyta, kad termo optinių koeficientų vertės randamos naudojant šiuos rinkinius ženkliai skiriasi. Rastos radialinio ir tangentinio šiluminio lūžio rodiklio pokyčio analitinės išraiškos YAG tipo mechaniškai izotropiniams kristalams, naudojant plokščių įtempių bei plokščių deformacijų artinius. Aptartos šiluminio lūžio rodiklio pokyčiui aprašyti literatūroje dažnai naudojamos vadinamosios termo optinės konstantos. Pašalinti nesusipratimai, susiję su fotoelastiniais koeficientais, ir rastos naujos jų išraiškos. Parodyta, kad šių koeficientų panaudojimas nėra toks naudingas, kaip dažnai manoma, ir kad esant skirtingiems kaupinimo pluoštams tai gali atvesti prie didelių šiluma indukuoto dvejopo lūžio skirtumų, palyginti su nuosekliai naudojamu plokščių deformacijų artiniu. Taip pat parodyta, kad šiluminio lęšio optinio stiprio vertės gali būti labai pervertinamos, kai plokščių įtempių formulės yra taikomos lazerio strypams su ilgiu, kelis kartus didesniu nei jų diametras.