Lithuanian Journal of Physics, Vol. **53**, No. 2, pp. 71–83 (2013)

BRANCHING RULES OF $\mathfrak{sp}(6) \downarrow \mathfrak{sp}(4) \times \mathfrak{sp}(2)$ AND BASES OF EIGENSTATES

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Received 26 September 2012; revised 16 November 2012; accepted 20 June 2013

An explicit formula describing the branching of representations of $\mathfrak{sp}(6)$ according to the reduction chain $\mathfrak{sp}(6) \downarrow \mathfrak{sp}(4) \times \mathfrak{sp}(2)$ is given. This allows to classify the multiplicity free reductions and, moreover, obtain the multiplicity for each $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ representation. We compare the method with the approach based on the theory of *S*-functions, pointing out the strengths and weaknesses of the explicit formula. The branching rule is used to construct an orthogonal basis of eigenstates for $\mathfrak{sp}(6)$, where degenerations are solved using a scalar instead of the standard missing label operator.

Keywords: representation of Lie groups, algebraic methods

PACS: 02.20.Qs, 02.40.Sv, 03.65.Fd

1. Introduction

In most physical applications where groups play an important role, we are usually encountered with the problem of determining how the representations of a symmetry group decompose as the sum of representations of some internal symmetry subgroup. For this reason, effective branching rules (BRs) for Lie algebra representations play an essential role in establishing approximate models and practical simulations, specifically in particle and nuclear physics, where the corresponding states of a system can be characterized by means of eigenvalues associated to invariant operators belonging to an adequate chain of symmetry groups and subgroups [1–5]. While often the labelling problem can be solved without major difficulties, sometimes the required reductions imply formidable computational obstructions, and beyond certain specifical cases an explicit general solution remains unknown [6–8].

In this context, extensive tabulations of branching rules, as well as specialized computer packages to determine branching rules have been developed through the years for various chains of Lie algebras/subalgebras of physical interest [9–12].

The problem of branching rules associated to semisimple (or reductive) Lie groups has been analysed in detail by different authors, providing a wealth of theoretical procedures useful for the explicit construction of states, like recursion relations, the boson realizations, the plethysm and tensor operator methods or the Gel'fand-Zetlin formalism that implicitly provide the branching rules with respect to the various subgroup chains [13–19]. Special mention for its effectiveness deserves the Littlewood expansion method in terms of *S*-functions [20] that has provided general formulae for the branching rules of many classical Lie groups. Variations of this procedure have enabled to obtain quite general expressions for the branching rules associated to maximal subgroups of Lie groups ([21, 22] and references therein), as well as to determine the multiplicities of the corresponding subgroup representations [21].

Concerning the Gel'fand-Zetlin patterns [23, 24], while these are naturally adapted to the unitary and orthogonal Lie algebras $\mathfrak{u}(N)$ and $\mathfrak{so}(N)$, they do not work properly for the symplectic group $Sp(2N)$, mainly due to the loss of quantum numbers in the reductions. The failure of the Gel'fand-Zetlin approach for this class led Zhelobenko to consider the branching problem for the Lie algebras $\mathfrak{sp}(2N)\downarrow\mathfrak{sp}(2N-2)\times\mathfrak{u}(1)$, whose solution was obtained by means of generalized patterns [25]. The analogous problem for the chain $\mathfrak{sp}(2N)\downarrow\mathfrak{sp}(2N-2)\times\mathfrak{sp}(2)$ was solved in a similar way in [26], where patterns were subjected to a system of inequalities that contained the branching rules in some implicit way. Although these solutions are not as systematic as the original Gel'fand-Zetlin patterns, due to degeneracy, they constitute a structurally important result. The main practical inconvenience of this approach lies in the fact that for a given representation of $\mathfrak{sp}(2N)$, all states must be computed explicitly in order to detect the branching rule and further determine whether the reduction is multiplicity free, i. e., if subgroup representations appear more than once or not. It must be mentioned that the BRs for the chain $\mathfrak{sp}(2N)\downarrow\mathfrak{sp}(2N-2)\times\mathfrak{sp}(2)$ were obtained in full generality in [21] by means of the Littlewood-Richardson formalism and identities satisfied by the infinite series of S-functions, leading to remarkably simple formulae that also described multiplicities.

The Lie algebras $\mathfrak{sp}(2N)$ and the corresponding non-compact forms appear in various physical models, where specially the cases $N=2,3$ have been shown to be of current interest in applications. For $N=2$ the chain $\mathfrak{sp}(4,\mathbb{R})\downarrow\mathfrak{sp}(2,\mathbb{R})\times\mathfrak{sp}(2,\mathbb{R})$ has been used in the study of light nuclei, more specifically in the simplification of the collective excitations of the $\mathfrak{sp}(6,\mathbb{R})\supset U(3)$ model [27, 28], albeit for the case of infinite dimensional (unitary) representations, while the chain $\mathfrak{sp}(6,\mathbb{R})\downarrow\mathfrak{sp}(4,\mathbb{R})\times\mathfrak{sp}(2,\mathbb{R})$ has recently been considered in the context of $\mathcal{N}=8$ supergravity truncation to $\mathcal{N}=2$ theories with scalar-vector minimal coupling [29][1], where branching rules are used to reduce kinematically the supergravity multiplets. These recent applications serve as motivation to the problem of obtaining branching rules in an explicit and consistent way.

[1] Actually in this frame the reduction chain is given by $E_{7(7)}\supset\mathfrak{sp}(6,\mathbb{R})\times G_{2(2)}\downarrow\mathfrak{sp}(4,\mathbb{R})\times\mathfrak{sp}(2,\mathbb{R})\times G_{2(2)}$.

In this work, as an alternative to other methods developed in the literature, specifically to the S-functions method, we construct an explicit formula for the branching rules of arbitrary irreducible representations for the chain $\mathfrak{sp}(6)\downarrow\mathfrak{sp}(4)\times\mathfrak{sp}(2)$. These formulae allow to solve some questions that are not immediate from the generic approach of [25] and [26], namely the determination of the multiplicity free representations and the exact multiplicity of each $\mathfrak{sp}(4)\times\mathfrak{sp}(2)$ representation appearing in the decomposition. In this context, we will compare our results with the approach by means of S-functions [20–22], and comment on the gains and losses of the direct method. As an application of the latter we will consider, in combination with scalar inequalities, the construction of orthogonal bases of states for commuting operators, avoiding the cumbersome step of determining an additional missing label operator. Multiple representations will be distinguished unambiguously by a scalar arising from the branching rules.

2. The branching rule $\mathfrak{sp}(6)\downarrow\mathfrak{sp}(4)\times\mathfrak{sp}(2)$

Following [25, 26], for the reduction chain $\mathfrak{sp}(6)\downarrow\mathfrak{sp}(4)\times\mathfrak{sp}(2)$ the branching rules for irreducible representations (IRREPs) are contained in the following generic pattern:

$$\left|\begin{matrix}\Omega_1^3 & & \Omega_2^3 & & \Omega_3^3 & \\ & \Gamma_1^3 & & \Gamma_2^3 & & h_3 \\ & & \Omega_1^2 & & \Omega_2^2 & \\ & & & \Gamma_1^2 & & h_2 \\ & & & & \Omega_1^1 & \\ & & & & & h_1\end{matrix}\right\rangle, \tag{1}$$

where the different entries satisfy the system of inequalities and identities($2\leq k\leq 3$):

$$\begin{gathered}\Omega_1^k\geq\Gamma_1^k\geq\Omega_2^k\geq\Gamma_2^k\geq\cdots\geq\Gamma_{k-1}^k\geq\Omega_k^k,\\ \Gamma_1^k\geq\Omega_1^{k-1}\geq\Gamma_2^k\geq\Omega_2^{k-1}\geq\cdots\geq\Gamma_{k-1}^k\geq\Omega_{k-1}^{k-1},\\ \sum_{p=a}^{k-1}\left(\Omega_{p+1}^k+\Omega_p^{k-1}\right)\geq\Gamma_a^k+2\sum_{p=a+1}^{k-1}\Gamma_p^k;\ a=1,\cdots,\ k-1,\\ \sigma_k=\sum_{p=1}^{k}\Omega_p^k+\sum_{p=1}^{k-1}\Omega_p^{k-1}-2\sum_{p=1}^{k-1}\Gamma_p^k,\\ h_k=-\sigma_k,-\sigma_k+2,\cdots,\sigma_k-2,\sigma_k.\end{gathered} \tag{2}$$

Given an IRREP $[\Omega_1^3,\Omega_2^3,\Omega_3^3]$ of $\mathfrak{sp}(6)$, equation (1) provides all possible states within this multiplet.

This implicit solution to the branching problem has the practical disadvantage of requiring the determination of all states for each fixed representation and can become quite cumbersome for representations of high dimension. It would therefore be desirable to further analyse the properties of these patterns in order to have an explicit formula giving the decomposition of an irreducible representation $[\Omega_1^3, \Omega_2^3, \Omega_3^3]$ of $\mathfrak{sp}(6)$ into a sum of irreducible representations of the subalgebra $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$, without being forced to compute the entries for the whole pattern (1). Such a formula would provide an effective tool to analyse those representations of $\mathfrak{sp}(6)$ that are multiplicity free, i. e., such that no representation of the subalgebra appears more than once, and furthermore provide the exact multiplicity for each component in the subalgebra. The objective of this Section is to determine such a formula and its consequences.

For $\mathfrak{sp}(6)$ the defining representation of dimension six decomposes as

$$[1,0,0] \downarrow (0)\,[1,0] + (1)[0,0]. \tag{3}$$

Writing down explicitly the six states, it is straightforward to verify that the $\mathfrak{sp}(4)\times\mathfrak{sp}(2)$ representations are completely characterized by the values of the pattern entries Ω_1^2, Ω_2^2, and σ_3. The problem of branching rules therefore reduces to determine the range of possible values for these entries when $[\Omega_1^3, \Omega_2^3, \Omega_3^3]$ are fixed.

For our explicit computations, it will be convenient to introduce the following notation: For $\lambda \geq \mu$ the representation $(m)[\lambda, \mu]$ of $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ will be denoted by the pattern $\left|\begin{matrix}\lambda\ \mu \\ m\end{matrix}\right\rangle$.

In the first instance, we extract the precise decomposition formula contained in the patterns (1):

Theorem 1. The BR for the reduction $\mathfrak{sp}(6) \downarrow \mathfrak{sp}(4) \times \mathfrak{sp}(2)$ and the IRREP $[k, l, m]$ is given by

$$[k,l,m]\downarrow \sum_{i=0}^{k-l}\sum_{j=0}^{l-m}\sum_{\beta=0}^{l-m+i-j}\sum_{\alpha=0}^{m-i}\left|\begin{matrix} m+j+\beta \quad i+j+\alpha \\ k-l-i+\alpha+\beta\end{matrix}\right\rangle + \sum_{\delta=0}^{m}\sum_{i=\delta+1}^{k-l}\sum_{j=0}^{l-m}\sum_{\gamma=0}^{l-m+\delta-j}\left|\begin{matrix} i+j+m-\delta+\gamma \quad j+\delta \\ k-l-i+\gamma\end{matrix}\right\rangle. \tag{4}$$

As observed before, the entries of the pattern (1) determining the representations appearing in the decomposition of $[\Omega_1^3, \Omega_2^3, \Omega_3^3] = [k, l, m]$ are Ω_1^2, Ω_2^2, and σ_3, the latter indicating the highest weight of the $\mathfrak{sp}(2)$ representation. Hence only those inequalities of (2) where these scalars appear will be relevant to determine the complete branching rule. The needed inequalities and identities are

$$\begin{aligned} &k \geq \Gamma_1^3 \geq l \geq \Gamma_2^3 \geq m, \\ &\Gamma_1^3 \geq \Omega_1^2 \geq \Gamma_2^3 \geq \Omega_2^2, \\ &l + m + \Omega_1^2 + \Omega_2^2 \geq \Gamma_1^3 + 2\Gamma_2^3, \\ &m + \Omega_2^2 \geq \Gamma_2^3, \\ &\sigma_3 = k + l + m + \Omega_1^2 + \Omega_2^2 - 2\left(\Gamma_1^3 + \Gamma_2^3\right). \end{aligned} \tag{5}$$

From the first inequality we can easily deduce the range for the auxiliary parameters Γ_1^3 and Γ_2^3:

$$\begin{aligned} &\Gamma_1^3 = 1, l+1, \ldots, k, \\ &\Gamma_1^2 = m, m+1, \ldots, l. \end{aligned}$$

In order to obtain the possible highest weights σ_3 of $\mathfrak{sp}(2)$, we must first compute, for each value of Γ_1^3 and Γ_2^3, the possible values of the highest weights Ω_1^2 and Ω_2^2 for $\mathfrak{sp}(4)$ representations. We thus proceed stepwise, by fixing the value of Γ_1^3 and Γ_2^3 and obtaining the representations of the subalgebra appearing for this particular choice.

Let $\Gamma_1^3 = l$ and $\Gamma_2^3 = m + j$, where $j = 0, \ldots, l - m$. In this case, the first of the inequalities of (5) is trivially satisfied, while the remaining simplify to

$$\begin{aligned} &l \geq \Omega_1^2 \geq m + j \geq \Omega_2^2, \\ &\Omega_1^2 + \Omega_2^2 \geq m + 2j, \\ &m + \Omega_2^2 \geq m + j, \\ &\sigma_3 = k - l - m - 2j + \Omega_1^2 + \Omega_2^2. \end{aligned}$$

As $m \geq 0$, the first and third inequalities imply that $l \geq \Omega_1^2 \geq m + j \geq \Omega_2^2 \geq j$. This enables us to determine the range of values possible for Ω_1^2 and Ω_2^2 that we specify as

$$\begin{aligned} &\Omega_1^2 = m + j + \beta,\ \beta = 0, \ldots, l - m - j, \\ &\Omega_2^2 = j + \alpha,\ \alpha = 0, \ldots, m. \end{aligned}$$

According to (5), the value of σ_3 is given by

$$\sigma_3 = k - l + \beta + \alpha.$$

It follows that for a fixed value of $j = 0, \ldots, m$, the IRREP $[k, l, m]$ contains the following $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ representations:

$$S_{[l,m+j]}=\sum_{\beta=0}^{l-m-j}\sum_{\alpha=0}^{m}\left|\begin{matrix} m+j+\beta & j+\alpha \\ k-l+\alpha+\beta & \end{matrix}\right\rangle. \quad (6)$$

Now we sum over j to obtain the partial sum

$$S_{[l]}=\sum_{j=0}^{l-m}\sum_{\beta=0}^{l-m-j}\sum_{\alpha=0}^{m}\left|\begin{matrix} m+j+\beta & j+\alpha \\ k-l+\alpha+\beta & \end{matrix}\right\rangle. \quad (7)$$

This sum specifies the $\mathfrak{sp}(4)\times\mathfrak{sp}(2)$ representations that we obtain for the value $\Gamma_1^3 = l$ and $\Gamma_2^3 = m, ..., l$. Following this procedure, the complete BR for $[k, l, m]$ will therefore arise from the sum over all possible values of Γ_1^3.

As the next case let $\Gamma_1^3 = l + 1$ and $\Gamma_2^3 = m + j$, where again $j = 0, ..., l - m$. The conditions to be satisfied here are

$$\begin{aligned} & l+1\geq\Omega_1^2\geq m+j\geq\Omega_2^2, \\ & \Omega_1^2+\Omega_2^2\geq m+2j+1, \\ & \Omega_2^2\geq j, \\ & \sigma_3=k-l-m-2j-2+\Omega_1^2+\Omega_2^2. \end{aligned} \quad (8)$$

At this point, two cases must be considered carefully. If $\Omega_2^2 = j$, then we easily deduce that $\Omega_1^2 \geq m + j + 1$, which means that Ω_1^2 cannot take the minimal value given by the first inequality of (8). We can write $\Omega_1^2 = m + j + 1 + \gamma$ with $\gamma = 0, ..., l - j - m$. This leads to the sum of patterns

$$S^{(2)}_{[l+1,m+j]}=\sum_{\gamma=0}^{l-m-j}\left|\begin{matrix} 1+m+j+\gamma & j \\ k-l-1+\gamma & \end{matrix}\right\rangle. \quad (9)$$

Observe in particular that the second entry of the patterns is always the same for fixed j. Now, if $\Omega_2^2 > j$, we write $\Omega_2^2 = 1 + j + \alpha$ with $\alpha = 0, ..., m - 1$. It follows from the remaining inequalities that $\Omega_1^2 = m + j + \beta$ for some $\beta = 0, ..., l - m + 1 - j$. The minimal value $j + m$ for Ω_1^2 is therefore possible. For a given fixed value of j we get the sum

$$S^{(1)}_{[l+1,m+j]}=\sum_{\beta=0}^{l-m-j}\sum_{\alpha=0}^{m}\left|\begin{matrix} m+j+\beta & 1+j+\alpha \\ k-l-1+\alpha+\beta & \end{matrix}\right\rangle. \quad (10)$$

Now we sum (9) and (10) over all possible values of $j = 0, ..., m$, leading us to the partial sum

$$\begin{aligned} S_{[l+1]}=&\sum_{j=0}^{l-m}\sum_{\beta=0}^{l-m-j}\sum_{\alpha=0}^{m}\left|\begin{matrix} m+j+\beta & 1+j+\alpha \\ k-l-1+\alpha+\beta & \end{matrix}\right\rangle + \\ &\sum_{j=0}^{l-m}\sum_{\beta=0}^{l-m-j}\left|\begin{matrix} 1+m+j+\gamma & j \\ k-l-1+\gamma & \end{matrix}\right\rangle. \end{aligned} \quad (11)$$

Formula (11) describes all $\mathfrak{sp}(4)\times\mathfrak{sp}(2)$ representations obtained for the values $\Gamma_1^3 = l + 1$ and $\Gamma_2^3 = m, ..., l$.

The general case works exactly like the last one, where two cases must be separated, according to whether Ω_1^2 can achieve the minimal possible value or not. Let $\Gamma_1^3 = l + i$ and $\Gamma_2^3 = m + j$, where $i = 0, ..., k - l$ and $j = 0, ..., l - m$. The first condition of (5) is satisfied, while the remaining constraints are given by

$$\begin{aligned} & l+i\geq\Omega_1^2\geq m+j\geq\Omega_2^2, \\ & \Omega_1^2+\Omega_2^2\geq m+i+2j, \\ & \Omega_2^2\geq j, \\ & \sigma_3=k-l-m-2j-2i+\Omega_1^2+\Omega_2^2. \end{aligned} \quad (12)$$

The separation of cases is as follows: if $\Omega_1^2 \geq m + j$ and the minimal value can be achieved, then by the second inequality we must have $\Omega_2^2 \geq i + j$. Taking $\Omega_1^2 \geq m + j + \beta$ for $\beta = 0, ..., l - m+1 - j$ and $\Omega_2^2 = i + j + \alpha$ with, $\alpha = 0,..., m - i$, for any fixed value of j we deduce the sum

$$S^{(1)}_{[l+i,m+j]}=\sum_{\beta=0}^{l-m-j}\sum_{\alpha=0}^{m}\left|\begin{matrix} m+j+\beta & i+j+\alpha \\ k-l-i+\alpha+\beta & \end{matrix}\right\rangle. \quad (13)$$

The second possibility arises when the value $\Omega_1^2 \geq m + j$ cannot be achieved because of the second inequality. In this case we have that $\Omega_2^2 = j + \delta$, where $0 \leq \delta \leq m$ and additionally $i \geq \delta + 1$. For these values we have $\Omega_1^2 = m + j + i - \delta + \gamma$ with $\gamma = 0, ..., l - j - m + \delta$, and summing these patterns over γ we get

$$S^{(2)}_{[l+i,m+j]}=\sum_{\gamma=0}^{l-m+\delta-j}\left|\begin{matrix} m+i+j-\delta+\gamma & j+\delta \\ k-l-i+\gamma & \end{matrix}\right\rangle. \quad (14)$$

Taking the sum over $j = 0,..., l - m$ and adding the terms (13) and (14) results in the partial sum

$$\begin{aligned} S_{[l+i]}=&\sum_{j=0}^{l-m}\sum_{\beta=0}^{l-m-j}\sum_{\alpha=0}^{m}\left|\begin{matrix} m+j+\beta & i+j+\alpha \\ k-l-i+\alpha+\beta & \end{matrix}\right\rangle + \\ &\sum_{j=0}^{l-m}\sum_{\gamma=0}^{l-m+\delta-j}\left|\begin{matrix} m+i+j-\delta+\gamma & j+\delta \\ k-l-i+\gamma & \end{matrix}\right\rangle. \end{aligned} \quad (15)$$

Like before, this formula provides the representations of the subalgebra that appear for the fixed value $\Gamma_1^3 = l + i$ and the recurring values $\Gamma_2^3 = m,..., l$.

The final step is to consider the sum over i of all partial sums obtained previously, covering thus the whole range of values of Γ_1^3 and Γ_2^3. The branching rule is therefore given by the finite sum

$$[k,l,m]\downarrow\sum_{i=0}^{k-l} S_{[l+i]},$$

which coincides with (4) as claimed. We observe that in the second series the index i begins with $\delta + 1$. This is a consequence of the constraints imposed in (14) on the values of Ω_1^2 and Ω_2^2.

The dimension of $[k, l, m]$ is easily recovered from the decomposition (4). Let us denote the dimensions of the patterns as

$$d^{k,l,m}_{\alpha,\beta,i,j} = \dim\left|\begin{matrix} m+j+\beta & i+j+\alpha \\ k-l-i+\alpha+\beta & \end{matrix}\right\rangle,$$

$$d^{k,l,m}_{i,j,\gamma,\delta} = \dim\left|\begin{matrix} m+i+j-\delta+\gamma & j+\delta \\ k-l-i+\gamma & \end{matrix}\right\rangle.$$

Summing up these dimensions for all values of the parameters we deduce

$$\dim[k,l,m] = \sum_{i=0}^{k-l}\sum_{j=0}^{l-m}\sum_{\beta=0}^{l-m+i-j}\sum_{\alpha=0}^{m-i} d^{k,l,m}_{\alpha,\beta,i,j} + \sum_{\delta=0}^{m}\sum_{i=\delta+1}^{k-l}\sum_{j=0}^{l-m}\sum_{\gamma=0}^{l-m+\delta-j} d^{k,l,m}_{i,j,\gamma,\delta} = \frac{1}{6!}(3+k)(1+k-l)(2+l)(5+k+l)(2+k-m)(1+l-m)\times(1+m)(4+k+m)(3+l+m). \tag{16}$$

A practical advantage of the explicit formula (4) over the patterns (1) is that we can now analyse whether the reduction of a $\mathfrak{sp}(6)$ representation is multiplicity free or not, and further, that we can predict the exact multiplicity of any component intervening in this decomposition.

Theorem 2. Let Γ be an IRREP $\mathfrak{sp}(6)$. Then Γ is multiplicity free in the reduction $\Gamma\downarrow\mathfrak{sp}(4)\oplus\mathfrak{sp}(2)$ only if Γ corresponds to one of the following cases:

(i) $\Gamma = [k, k, m]$, $k \geq m$

(ii) $\Gamma = [k, l, l]$, $k \geq l$

(iii) $\Gamma = [k, l, 0]$, $k \geq l$

We divide the proof into two parts. First we show that the previous representations are actually multiplicity free in the reduction to the subalgebra $\mathfrak{sp}(4)\oplus\mathfrak{sp}(2)$, and then we show that these exhaust the possibilities.

Consider first the IRREP $[k, k, m]$ with $k \geq m$. In this case the reduction reads

$$[k,k,m]\downarrow\sum_{j=0}^{k-m}\sum_{\beta=0}^{k-m-j}\sum_{\alpha=0}^{m}\left|\begin{matrix} m+j+\beta & j+\alpha \\ \alpha+\beta & \end{matrix}\right\rangle. \tag{17}$$

We observe that the second sum in (4) is identically zero because it requires i to be greater than or equal to $1 + \delta$. If two (or more) terms were repeated, from the coincidence of two patterns

$$\left|\begin{matrix} m+j+\beta & j+\alpha \\ \alpha+\beta & \end{matrix}\right\rangle = \left|\begin{matrix} m+j'+\beta' & j'+\alpha' \\ \alpha'+\beta' & \end{matrix}\right\rangle$$

we deduce the following identities:

$$j+\beta = j'+\beta',\ j+\alpha = j'+\alpha,\ \alpha+\beta = \alpha'+\beta'.$$

Using the first two we obtain that $\beta = (j' - j) + \beta'$ and $\alpha = (j' - j) + \alpha'$ and inserting this into the third equality and simplifying the expression we get $j' - j = 0$, hence $j = j'$ and $\alpha = \alpha'$, $\beta = \beta'$. Therefore, all patterns intervening in the decomposition (17) are different.

If $\Gamma = [k, l, l]$, by formula (4) we obtain the decomposition

$$[k,l,l]\downarrow\sum_{i=0}^{k-l}\sum_{\beta=0}^{i}\sum_{\alpha=0}^{l-i}\left|\begin{matrix} l+\beta & i+\alpha \\ k-l-i+\alpha+\beta & \end{matrix}\right\rangle + \sum_{\delta=0}^{l}\sum_{i=\delta+1}^{k-l}\sum_{\gamma=0}^{\delta}\left|\begin{matrix} i+l+\gamma-\delta & \delta \\ k-l-i+\gamma & \end{matrix}\right\rangle. \tag{18}$$

If two patterns of the first sum coincide, then

$$\left|\begin{matrix} l+\beta & i+\alpha \\ k-l-i+\alpha+\beta & \end{matrix}\right\rangle = \left|\begin{matrix} l+\beta' & i'+\alpha' \\ k-l-i'+\alpha'+\beta' & \end{matrix}\right\rangle.$$

Clearly $\beta = \beta'$ and $\alpha = i' - i + \alpha'$. From the third entry we further have that $\alpha = \alpha' - i' + i$. The difference of these two expressions in α leads to $i = i'$, and in consequence $\alpha = \alpha'$. and no repetition is possible. Now suppose that in the second sum there is a repetition

$$\left|\begin{matrix} i+l+\gamma-\delta & \delta \\ k-l-i+\gamma & \end{matrix}\right\rangle = \left|\begin{matrix} i'+l+\gamma'-\delta' & \delta' \\ k-l-i'+\gamma' & \end{matrix}\right\rangle.$$

It is immediate that $\delta = \delta'$, $\gamma = i = \gamma' + i'$, and $\gamma - i = \gamma' - i'$. Solving the two last equalities for γ' and comparing them we are led to $i = i'$, hence to $\gamma = \gamma'$, showing that no repeated patterns appear. It remains to exclude the coincidence of a pattern in the first sum and a pattern in the second:

$$\left|\begin{matrix} l+\beta & i+\alpha \\ k-l-i+\alpha+\beta \end{matrix}\right\rangle = \left|\begin{matrix} i'+l+\gamma-\delta & \delta \\ k-l-i'+\gamma \end{matrix}\right\rangle.$$

Here $\delta = i + \alpha$, $\beta = \gamma + i' - \delta$, and $\alpha + \beta - i = \gamma - i'$. Inserting the value of δ in the second equality and solving for γ we obtain that $\gamma = \alpha + \beta + i - i'$, while from the third one we deduce that $\gamma = \alpha + \beta - i + i'$. From a comparison of these two expressions it follows at once that $i = i'$. Now i' is further constrained by $i' \geq \delta + 1 = i + \alpha + 1$ (see (4)); therefore, we obtain the inequality

$$i = i' \geq i + \alpha + 1.$$

As $i \geq 0$, this would imply that $\alpha + 1 \leq 0$, which is impossible since $\alpha \geq 0$. This shows that a coincidence of patterns in the first and second sums cannot happen.

Finally, for the IRREP $[k, l, 0]$ the decomposition has the form

$$[k,l,0]\downarrow \sum_{i=0}^{k-l}\sum_{j=0}^{l}\sum_{\beta=0}^{l+i-j}\sum_{\alpha=0}^{-i}\left|\begin{matrix} j+\beta & i+j+\alpha \\ k-l-i+\alpha+\beta \end{matrix}\right\rangle + \sum_{i=1}^{k-l}\sum_{j=0}^{l}\sum_{\gamma=0}^{l-j}\left|\begin{matrix} i+j+\gamma & j \\ k-l-i+\gamma \end{matrix}\right\rangle. \quad (19)$$

From the first sum we see that, since $\alpha \geq 0$, we must have $i = \alpha = 0$. Hence the first sum reduces to

$$\sum_{i=0}^{k-l}\sum_{j=0}^{l}\sum_{\beta=0}^{l-j}\left|\begin{matrix} j+\beta & j \\ k-l+\beta \end{matrix}\right\rangle.$$

From this expression it is immediate that no repetition of patterns is possible for different values of j. If

$$\left|\begin{matrix} i+j+\gamma & j \\ k-l-i+\gamma \end{matrix}\right\rangle = \left|\begin{matrix} i'+j'+\gamma' & j' \\ k-l-i'+\gamma' \end{matrix}\right\rangle,$$

then $j = j'$, $\gamma + i = \gamma' + i'$, and $\gamma - i = \gamma' - i'$. As before, this leads to $\gamma = \gamma'$ and $i = i'$, thus no repetition is possible. The last possibility is that one pattern in the first sum and one in the second sum coincide:

$$\left|\begin{matrix} j+\beta & j \\ k-l+\beta \end{matrix}\right\rangle = \left|\begin{matrix} i'+j'+\gamma & j' \\ k-l-i'+\gamma \end{matrix}\right\rangle.$$

Comparing the entries we obtain that $j = j'$, $\beta = \gamma + i'$, and $\beta = \gamma - i'$, and this implies that $i' = 0$, which is excluded since $i' \geq 1$.

This finishes the proof that the three types of IRREPs are multiplicity free. It remains to show that they are the only ones having this property. To this intent, it will be enough to obtain the conditions for which patterns of the first sum of decomposition (4) appear more than once. Suppose that the following equality holds:

$$\left|\begin{matrix} m+j+\beta & i+j+\alpha \\ k-l-i+\alpha+\beta \end{matrix}\right\rangle = \left|\begin{matrix} m+j'+\beta' & i'+j'+\alpha' \\ k-l-i'+\alpha'+\beta' \end{matrix}\right\rangle.$$

Then the following identities hold:

$$j + \beta = j' - \beta',\ i + j + \alpha = i' + j' + \alpha',$$
$$-i + \alpha + \beta = -i' + \alpha' + \beta'.$$

Inserting $\beta = j' - j + \beta'$ into the third condition and adding it to the second one we obtain that $\alpha = \alpha'$. In particular this equality implies that

$$i + j = i' + \beta'. \quad (20)$$

This equality, jointly with $\beta = j' - j + \beta'$, expresses the essential condition for multiplicity. Indeed, for any integer $n \geq 1$ the partitions of n as the sum of two non-negative integers[2] opens the possibility for the existence of repeated patterns with pairwise different entries in $(\beta, i, j) \neq (\beta', i', j')$, i. e. we would have specifically

$$\left|\begin{matrix} m+j+(\beta'+j'-j) & i+j+\alpha \\ k-l-i+\alpha+(\beta'+j'-j) \end{matrix}\right\rangle = \left|\begin{matrix} m+j'+\beta' & i'+j'+\alpha \\ k-l-i'+\alpha+\beta' \end{matrix}\right\rangle.$$

As a consequence, the only possibility to avoid multiplicity is that in equation (20) we can ensure the identities $i = i'$ and $j = j'$. This can only happen if either the indices i, i' or j, j' take only one and the same value. From (4) we see that the range for the different indices is given by

$$0 \leq i, i' \leq k - l;\ 0 \leq j, j' \leq l - m;\ 0 \leq \beta, \beta' \leq m - i.$$

In view of this, if i, i' and j, j' can only take one value, then one of the following three possibilities must occur:

(i) If $k = l$, then $i = i' = 0$ and by (20) we get $j = j'$, hence $\beta = \beta'$. We have seen in the first part that this case corresponds to the (multiplicity free) representations $[k, k, m]$.

[2] We include the possibility that either i or j is zero.

(ii) If $l = m$, then it is immediate that $j = j' = 0$ and $\beta = \beta'$. This corresponds to the IRREPs $[k, l, l]$.

(iii) If $m = 0$, then $0 \le \beta \le -i$ and $0 \le \beta' \le -i'$. As both i, i' are non-negative, this would imply that $i = i' = 0$, and thus $\beta = \beta'$ and $j = j'$.[3] For this case we recover the IRREPs $[k, l, 0]$.

For any other choice of values k, l, m some pattern in the first sum (4) will be repeated, which makes it unnecessary to inspect either the second sum or coincidence of patterns in both parts. This means that this analysis proves that the only representations of $\mathfrak{sp}(6)$ to be multiplicity free with respect to the reduction $\mathfrak{sp}(6) \downarrow \mathfrak{sp}(4) \times \mathfrak{sp}(2)$ are $[k, k, l]$, $[k, l, l]$, and $[k, l, 0]$.

The previous result can be proved alternatively, albeit with more cumbersome computations, using directly the inequalities (2) associated to the states (1). In order to determine the precise multiplicity of a pattern $\left|\begin{matrix}\lambda\ \mu\\ m\end{matrix}\right\rangle$ in the decomposition of a given IRREP $[k, l, m]$ of $\mathfrak{sp}(6)$, this direct approach can be more convenient.

Pattern multiplicity criterion. If the pattern $\left|\begin{matrix}\lambda\ \mu\\ \nu\end{matrix}\right\rangle$ has multiplicity q_0 in the decomposition (4) of $[k, l, m]$, then the scalar equation

$$\frac{1}{2}\left(k+l+m+\lambda+\mu-\nu\right)=\Gamma_1^3+\Gamma_2^3 \tag{21}$$

has exactly q_0 solutions (Γ_1^3, Γ_2^3), where $k \ge \Gamma_1^3 \ge l$ and $l \ge \Gamma_2^3 \ge m$. A computational convenience of (21) is that it can be solved with total independence of the patterns (1).

Suppose that the irreducible representation $(\nu)\,[\lambda, \mu]$ of $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ (corresponding to the previous pattern) appears more than once in decomposition (4) and let q_0 denote this multiplicity. In this case, the inequalities (2) to be satisfied are

$$\begin{array}{ll} k\ge\Gamma_1^3\ge l\ge\Gamma_2^3\ge m, & \Gamma_1^3\ge\lambda\ge\Gamma_2^3\ge\mu,\\ l+m+\lambda+\mu\ge\Gamma_1^3+2\Gamma_2^3, & m+\mu\ge\Gamma_2^3,\\ \mu+\Omega_1^1\ge\Gamma_1^2\ge\Omega_1^1, & \lambda\ge\Gamma_1^2\ge\mu. \end{array} \tag{22}$$

We observe that Ω_1^1 and Γ_1^2 are constrained by λ and μ alone, which means that these quantities will only have the effect of distinguishing states within each $\mathfrak{sp}(4)$-representation $[\lambda, \mu]$. Therefore only the remaining scalars Γ_1^3 and Γ_2^3 can refer to the multiplicity of such representations. As the value ν is also fixed, the possible values of Γ_1^3 and Γ_2^3 will also have to satisfy the equation

$$\nu = k + l + m + \lambda + \mu - 2(\Gamma_1^3, \Gamma_2^3). \tag{23}$$

It follows at once that the number of different copies of $(\nu)\,[\lambda, \mu]$ contained in $[k, l, m]$ is completely determined by the pairs (Γ_1^3, Γ_2^3) satisfying this equation.

As an example how this fact can be used to separate degeneracies in formula (4), we consider the lowest dimensional IRREP of $\mathfrak{sp}(6)$ exhibiting multiplicities. Using (4), the representation [3, 2, 1] of dimension 512 branches as

$$\begin{aligned}[3,2,1]\downarrow &\left|\begin{matrix}1\,0\\1\end{matrix}\right\rangle+\left|\begin{matrix}1\,1\\0\end{matrix}\right\rangle+\left|\begin{matrix}1\,1\\2\end{matrix}\right\rangle+\left|\begin{matrix}2\,0\\0\end{matrix}\right\rangle+\left|\begin{matrix}2\,0\\2\end{matrix}\right\rangle+2\left|\begin{matrix}2\,1\\1\end{matrix}\right\rangle+\\ &\left|\begin{matrix}2\,1\\3\end{matrix}\right\rangle+\left|\begin{matrix}2\,2\\0\end{matrix}\right\rangle+\left|\begin{matrix}2\,2\\2\end{matrix}\right\rangle+\left|\begin{matrix}3\,0\\1\end{matrix}\right\rangle+\left|\begin{matrix}3\,1\\0\end{matrix}\right\rangle+\left|\begin{matrix}3\,1\\2\end{matrix}\right\rangle+\left|\begin{matrix}3\,2\\1\end{matrix}\right\rangle.\end{aligned} \tag{24}$$

The representation (1) [2,1] of the subalgebra appears twice. From equation (21) we see that the only solutions to equation (23) with $\lambda = 2$, $\mu = 1$, and $\nu = 1$ are $(\Gamma_1^3, \Gamma_2^3) = \{(3,1), (2,2)\}$. Therefore, taking as label $\alpha = \Gamma_1^3$, both copies are properly distinguished.

The multiplicity criterion has a practical consequence, namely, the separation of representations of $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ having multiplicity greater than one in $[k, l, m]$ by the scalar values of Γ_1^3 or Γ_2^3.

2.1. Merits and demerits of the direct combinatorial approach

Formula (4) mainly constitutes a refinement of the patterns considered in [25,26]. It is conceivable to develop similar formulae for any fixed $N > 3$, although this requires first to solve the cases $3 \le m \le N - 1$. This means that there is no straightforward possibility of describing the generic case by a simple formula. This fact substantiates the main drawback of the procedure presented here, in contrast with the expansion method based on *S*-functions and their operations, where the generic case is described in a remarkably simple form in terms of positive terms [21, 22]. In

[3] It should be remarked that $m = 0$ does not imply that i can only take the value $i = 0$, but that it must be zero whenever the multiplicity condition (20) is imposed.

addition, the use of the series of *S*-functions and Littlewood-Richardson rules points out that the branching rule problem presents analogous features and similarities for different reduction chains, allowing to compare apparently different chains of Lie groups and extract information concerning multiplicities. Such relations remain certainly unnoticed when using other more direct procedures, as ours.

For the case under scrutiny in this work, it follows from the results in [21] that the multiplicity of the IRREP $(\nu)[\lambda,\mu]$ of $\mathfrak{sp}(4)\times\mathfrak{sp}(2)$ in the IRREP $[k,l,m]$ of $\mathfrak{sp}(6)$ coincides with the multiplicity in the following embeddings:

(i) multiplicity of the IRREP $[\lambda,\mu]\times[\nu+\Delta,\Delta]$ of $\mathfrak{u}(2\alpha+2)\times\mathfrak{u}(2)$ in the IRREP $[k,l,m]$ of $\mathfrak{u}(2\alpha+4)$ for $\alpha=0,1$,

(ii) multiplicity of the IRREP $[k,l,m]$ of $\mathfrak{u}(3)$ in the IRREP $[\lambda,\mu]\otimes[\nu,\Delta]$ of $\mathfrak{u}(3)\otimes\mathfrak{u}(3)$[4],

where $\Delta=\frac{1}{2}\left(k+l+m-\lambda-\mu-\nu\right)$. In view of these relations, it is clear that the application of the character theory goes far beyond the analysis of a specific reduction chain, as it also provides an insight into the BR of other Lie groups.

As to advantages of the ansatz proposed in this work, these are mainly of computational nature. While the *S*-function method requires the manipulation of Young diagrams as well as the application of simplification rules, the task of obtaining the BR may be laborious for high dimensional representations. In contrast, formula (4) only requires evaluation of a finite sum, and no further simplification or manipulation is required.

Another positive aspect of the direct approach is the possibility of solving questions that are far from being trivially handled with when considering the *S*-functions, e. g. the obtainment of those IRREPs $[k,l,m]$ of $\mathfrak{sp}(6)$ that contain a (fixed) irreducible representation $(\nu)\,[\lambda,\mu]$ of $\mathfrak{sp}(4)\times\mathfrak{sp}(2)$ as well as determining the maximal possible multiplicity.

Proposition 1 Let $R=(\nu)\,[\lambda,\mu]$ be a fixed IRREP of $\mathfrak{sp}(4)\times\mathfrak{sp}'(2)$. Then R appears in the decomposition of the following irreducible representations $[k,l,m]$ of $\mathfrak{sp}(6)$:

(i) For $\mu>m$:

$$[k,l,m]=[l+m+\lambda-\mu-\nu+2(b-a),l,m], \quad (25)$$

where $l\geq\mu$, $\lambda\geq m$, $0\leq b\leq\nu$, and

$0\leq a\leq\min\{l-\mu,\lambda-\mu,m\}$, $l+m+b\geq\mu+\nu+2a$.

The multiplicity q_0 of R in $[k,l,m]$ is bounded by $q_0\leq 1+\min\{l-\mu,\lambda-\mu,m\}$.

(ii) For $\mu\leq m$:

$$[k,l,m]=[l-m+\lambda+\mu-\nu+2(b-a),l,m], \quad (26)$$

where $l\geq\mu$, $\lambda\geq m$, $0\leq b\leq\nu$, and $0\leq a\leq\min\{l-m,\lambda-m,\mu\}$, $\lambda+\mu+b\geq m+\nu+2a$.

The multiplicity q_0 of R in $[k,l,m]$ is bounded by $q_0\leq 1+\min\{l-m,\lambda-m,\mu\}$.

Proof. We use identity (21) and the first four inequalities of (22). From $k\geq\Gamma_1^3\geq l\geq\Gamma_2^3\geq m$ and $\Gamma_1^3\geq\lambda\geq\Gamma_2^3\geq\mu$ we easily deduce that $l\geq\mu$ and $\lambda\geq m$, as well as $\Gamma_2^3\geq\max\{\mu,m\}$. We distinguish the two possible cases. If $\mu>m$, then $\Gamma_2^3=\mu+a$ for some $\alpha\geq 0$. Inserting it into (21) and (22) and simplifying the expressions lead to $k=2\Gamma_1^3-l-m-\lambda+\mu+\nu+2a$ and the inequalities

$$\begin{array}{ll} k\geq\Gamma_1^3\geq l\geq\mu+a, & \Gamma_1^3\geq\lambda\geq\mu+a,\\ l+m+\lambda-\mu-2a\geq\Gamma_1^3, & \mu\geq a. \end{array} \quad (27)$$

This implies that $a\leq\min\{l-\mu,\lambda-\mu,m\}$. In particular, equation (27) shows that the multiplicity q_0 of $(\nu)\,[\lambda,\mu]$ can never exceed $1+\min\{l-\mu,\lambda-\mu,m\}$. Now, as $k\geq\Gamma_1^3$ holds, combining the third inequality of (27) with the expression of k we further infer that

$$\Gamma_1^3\geq l+m+\lambda-\mu-\nu-2a.$$

It follows that $\Gamma_1^3=l+m+\lambda-\mu+\nu+2a+b$ for some $0\leq b\leq\nu$. Now, for each fixed value of $0\leq a\leq\min\{l-\mu,\lambda-\mu,m\}$ and b that satisfy the inequality $\lambda+\mu+b\geq m+\nu+2a$ the conditions (22) are satisfied, and (21) provides the value of k. This shows that R is contained in the representation $[l+m+\lambda-\mu-\nu+2(b-a),l,m]$.

We omit the detailed proof for the case $\mu\leq m$, as it is completely analogous to the previous one. ▪

Corollary 1. Let $q_0\geq 2$. Then $[3q_0-3,2q_0-2,q_0-1]$ is the lowest dimensional irreducible representation of $\mathfrak{sp}(6)$ such that the reduction

[4] In this context, we observe that a closed formula for the tensor products of representations of $SU(3)$was developed in [30].

$\mathfrak{sp}(6) \downarrow \mathfrak{sp}(4) \times \mathfrak{sp}(2)$ contains some representation $(\nu)\,[\lambda, \mu]$ having exactly multiplicity q_0.

Proof. Suppose that $(\nu)\,[\lambda, \mu]$ has exactly multiplicity q_0 in $[k, l, m]$. According to the previous result, two possibilities are given, according to whether $\mu \le m$ or $\mu > m$.

(i) Let $\mu \le m$. The upper bound for the multiplicity is given by $q_0 \le \min\{l - m, \lambda - m, \mu\} + 1$, implying that

$$l - m \ge q_0 - 1, \lambda - m \ge q_0 - 1, \mu \ge q_0 - 1.$$

In order to get the representation $[k, l, m]$ of the lowest possible dimension, let us put $m = \mu = q_0 - 1$, $l = m + q_0 - 1 = 2q_0 - 2$, and $\lambda = l$. Further let $\Gamma_2^3 = q_0 - 1 + a$, where $0 \le a \le q_0 - 1$. Inserting these values into (21) and (22) leads to the conditions

$$k \ge \Gamma_1^3 \ge 2q_0 - 2, 4q_0 - 4 - 2a \ge \Gamma_1^3,$$

where $k = 2\Gamma_1^3 + 2a + 4 - 4q_0 + \nu$. Since the value of k must remain the same for any $0 \le a \le q_0 - 1$, the equation (21) has exactly q_0 solutions if and only if

$$(\Gamma_1^3, \Gamma_2^3) = (3q_0 - 3 - a, q_0 - 1 + a), 0 \le a \le q_0 - 1.$$

It follows that $k = q_0 - 2 = \nu$, and the condition $k \ge \Gamma_1^3$ implies that $\nu \ge q_0 - 1$. The minimal value of k is given for $\nu = q_0 - 1$, and we obtain the IRREP $[3q_0 - 3, 2q_0 - 2, q_0 - 1]$. In this case, the representation of $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ having multiplicity q_0 is $(q_0 - 1)\,[2q_0 - 2, q_0 - 1]$.

(ii) For the case $\mu > m$ the reasoning is very similar to the previous one, and always choosing the minimal possible values $\lambda = l = 2q_0 - 1$, $\mu = m + 1 = q_0$, $\nu = q_0 - 1$ we obtain the IRREP $[3q_0 - 2, 2q_0 - 1, q_0 - 1]$ of $\mathfrak{sp}(6)$ containing the representation $(q_0 - 1)\,[2q_0 - 1, q_0]$ with multiplicity q_0.

Using the dimension formula (16) it follows that $q_0^9 = \dim[3q_0 - 3, 2q_0 - 2, q_0 - 1] < \dim\,[3q_0 - 2, 2q_0 - 1, q_0 - 1]$ for all q_0, showing that the former representation is the lowest dimensional exhibiting multiplicity q_0. ▪

Consequence. Let $a \ge 2$ and $\Lambda = [k, l, m]$ an irreducible representation of $\mathfrak{sp}(6)$. If $\dim\,[k, l, m] < a^9$, then the maximal multiplicity of an IRREP $(\nu)\,[\lambda, \mu]$ of $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ in Λ is $a - 1$.

As an example of the preceding criteria, let us consider the representation $R = \left|{}^{7\,2}_{\;2}\right\rangle$. The application of proposition 1 provides all the 16 types of IRREPs of $\mathfrak{sp}(6)$ whose reduction to $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ contain R:

- $[l + 7, l, 0]$, $[l + 6, l, 1]$, $[l + 8, l, 1]$ for $l \ge 2$;
- $[l + 5, l, 0]$, $[l + 4, l, 1]$ for $l \ge 3$;
- $[l + 3, l, 0]$ for $l \ge 4$;
- $[l + 2, l, 1]$ for $l \ge 5$;
- $[l - m + 7, l, m]$, $[l - m + 9, l, m]$, $[l - m + 11, l, m]$ for $l \ge m$ and $2 \le m \le 7$;
- $[l - m + 5, l, m]$ for $l \ge m + 1$ and $2 \le m \le 5$;
- $[l - m + 7, l, m]$, $[l - m + 9, l, m]$ for $l \ge m + 1$ and $2 \le m \le 6$;
- $[l - m + 3, l, m]$ for $l \ge m + 2$ and $2 \le m \le 3$;
- $[l - m + 5, l, m]$ for $l \ge m + 2$ and $2 \le m \le 4$;
- $[l - m + 7, l, m]$ for $l \ge m + 2$ and $2 \le m \le 5$.

In addition, it follows that the multiplicity of R in $[k, l, m]$ never exceeds 3.

3. Application to orthogonal bases of states

It is well known that for any semisimple Lie algebra $\mathfrak{s}$ of rank l there exist $\mathcal{N}(\mathfrak{s}) = l$ functionally independent Casimir operators and that eigenvalues of these label irreducible representations of s [31]. Racah pointed out that in general the Casimir operators and Cartan generators are not sufficient to completely characterize the states within a representation and that the total number of internal labels required is given by[5]

$$i = \frac{1}{2}(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s})). \tag{28}$$

If we use a subalgebra $\mathfrak{s}' \subset \mathfrak{s}$ to label the basis states, a similar lack of a complete set of labelling operators is observed. In this case the subgroup provides $\frac{1}{2}(\dim \mathfrak{s}' + \mathcal{N}(\mathfrak{s}')) - l_0$ labels, where l_0 is the number of invariants of $\mathfrak{s}$ that depend only on generators of the subalgebra $\mathfrak{s}'$ [6]. Additional

$$n = \frac{1}{2}\left(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s}) - \dim \mathfrak{s}' - \mathcal{N}(\mathfrak{s}')\right) + l_0 \tag{29}$$

[5] As the eigenvalues of the Casimir operators are the same for all states, we can skip them whenever the IRREP of $\mathfrak{s}$ is fixed.

operators, called missing label operators or subgroup scalars, are needed to separate multiplicities of IRREPs of $\mathfrak{s}'$. Supposed that these operators are taken in Hermitean form, they can be simultaneously diagonalized and hence any state of the representation $[k, l, m]$ will be characterized by the eigenvalues of these operators [32].

The missing label problem (MLP) for the chain $\mathfrak{sp}(6) \supset \mathfrak{sp}(4) \times \mathfrak{sp}(2)$ has been analysed from various different perspectives [7, 33]. However, for the practical choice of the missing label operator there is no natural candidate. The simplest suitable subgroup scalar Θ to separate degeneracies within a representation $[k, l, m]$ would have degree six in the generators, which makes its diagonalization a difficult practical problem, as well as its numercial evaluation. We will instead use the criterion (21) derived from the branching rule to circumvent this point.

To construct a basis of eigenstates for $\mathfrak{sp}(6)$ representations in a $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ basis, we need to find $i = 9$ internal commuting operators that in addition commute with all generators of the subalgebra. The "external" operators correspond to the Casimir operators of $\mathfrak{sp}(6)$. As the reduction is not multiplicity free, the previous formula (29) indicates that in general $n = 1$ missing label operator is sufficient to separate degeneracies. However, as we have seen, using equation (23) we can skip this operator, as repeated IRREPs of the subalgebra are distinguished by the values of Γ_1^3 and Γ_2^3.

It is convenient to use the Racah realization of $\mathfrak{sp}(6)$ to explicitly construct these operators [31]. We consider the Lie algebra generators $X_{i,j}$ with $-3 \le i, j \le 3$ satisfying the condition

$$X_{i,j} + \varepsilon_i \varepsilon_j X_{-j,-i} = 0, \qquad (30)$$

where $\varepsilon_i = \mathrm{sgn}(i)$. Over this basis, the brackets are given by

$$[X_{i,j}, X_{k,l}] = \delta_{jk} X_{il} - \delta_{il} X_{kj} + \varepsilon_i \varepsilon_j \delta_{j,-l} X_{k,-i} - \varepsilon_i \varepsilon_j \delta_{i,-k} X_{-j,l}, \qquad (31)$$

where $-3 \le i, j, k, l \le 3$. It is clear that the operators $X_{i,j}$ for which $-2 \le i, j \le 2$, jointly with $X_{3,3}$, $X_{-3,3}$, and $X_{3,-3}$ generate the subalgebra $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$. We further have that $h_i = X_{i,i}$ generate the Cartan subalgebra of $\mathfrak{sp}(6)$.

With this basis, a suitable choice for the commuting operators is given by [32]:

• The three Casimir operators I_2, I_4, I_6 of $\mathfrak{sp}(6)$. They have the same values for all states within the IRREP $[k, l, m]$.

• The three Cartan generators h_1, h_2, h_3 of $\mathfrak{sp}(6)$. The last one corresponds to the Cartan subalgebra of $\mathfrak{sp}(2)$, while the two first generate also the Cartan subalgebra of $\mathfrak{sp}(4)$.

• The three Casimir operators C_2, C_4, and C_2' of $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$. These will have the same values for all states belonging to the irreducible representation $(\nu)\,[\lambda, \mu]$ associated to the pattern $\left| \begin{matrix} \lambda\ \mu \\ \nu \end{matrix} \right\rangle$.

• The quadratic operators C_{21}, C_{22} obtained from the further reduction chain $\mathfrak{sp}(4) \supset \mathfrak{sp}(2) \times \mathfrak{sp}(2)$. These operators separate the states within each $\mathfrak{sp}(4)$ IRREP obtained in (4).

• The missing label operator Θ.

As we have observed previously, we can construct an orthogonal basis of states for arbitrary representations of $\mathfrak{sp}(6)$ without a necessity of computing explicitly the difficult to diagonalize missing label operator Θ. We skip this subgroup scalar, and as additional label to separate representations of the subalgebra we use the numerical value of Γ_1^3 (or Γ_2^3) obtained from equation (21).

Let ρ_2, ρ_4, ρ_6 be the eigenvalues of I_2, I_4, I_6. The value of ρ_2 can be easily computed, while for the computation of ρ_4 and ρ_6 it is convenient to use the so-called Okubo formula [34]. We give the explicit values for ρ_2, ρ_4, while we skip the expression for ρ_6 because of its length (83 terms):

$$\begin{aligned} \rho_2 &= k^2 + l^2 + m^2 + 6k + 4l + 4m, \\ \rho_4 &= 5k^4 + 5l^4 + 5m^2 + 3k^2l^2 + 3k^2m^2 + 3l^2m^2 + 60k^3 \\ &\quad + 40l^3 + 20m^{3+} + 12k^2l + 6k^2m + 18kl^2 + 18km^2 \\ &\quad + 6l^2m + 12lm^2 + 221k^2 + 86l^2 + 5m^2 + 72kl \\ &\quad + 36km + 24lm + 126k + 24l - 30m. \end{aligned} \qquad (32)$$

Further let ξ_2, ξ_4 and ξ_{21} denote the eigenvalues of C_2, C_4, and C_2' for a representation $(\nu)\,[\lambda, \mu]$. Their explicit expression in the basis above of $\mathfrak{sp}(6)$ is

$$\begin{aligned} \xi_2 &= \nu\,(4+\lambda) + \mu\,(2+\mu), \\ \xi_4 &= \frac{5}{2}\mu\,(2+\mu) + \lambda\,(1+4\lambda)\left(4\mu^2 + 8\mu - 2\right), \\ \xi_{21} &= \nu\,(2+\nu). \end{aligned} \qquad (33)$$

In order to determine the eigenvalues of the internal operators θ_{21} and θ_{22}, corresponding to the Casimir operators C_{21}, C_{22}, we must also know how the $\mathfrak{sp}(4)$ representation $[\lambda\mu]$ branches when reduced to the subalgebra $\mathfrak{sp}(2) \times \mathfrak{sp}(2)$. In this case the formula reads:

$$[\lambda\mu]\downarrow\sum_{s=0}^{\mu}\sum_{r=0}^{\lambda-\mu}(\lambda-r-s)(\mu+r-s). \qquad (34)$$

Hence, for each fixed values of r and s the corresponding eigenvalues are

$$\theta_{21}=(\lambda-r-s)(\lambda-r-s+2),$$
$$\theta_{22}=(\mu+r-s)(\mu+r-s+2). \qquad (35)$$

Putting together the previous eigenvalues, it follows that the orthogonal basis of $[k, l, m]$ is determined by the eigenstates

$$|\rho_1\ \rho_2\ \rho_3;\ \xi_2\ \ \xi_4\ \ \xi_{21};\ \theta_{21}\ \theta_{22};\ h_1\ \ h_2\ h_3;\ \alpha\rangle, \qquad (36)$$

where α denotes the scalar(s) Γ_1^3 obtained from equation (23).

To illustrate the use of (36) we give the complete basis of eigenstates for the IRREP [3, 2, 1] in Table 1. For each pattern appearing in the decomposition (24) the eigenvalues and the number of different states provided by them are given.

Table 1. Basis of eigenstates for [3, 2, 1] in $d = 512$*.

Pattern	Eigenstates	Constraints
$\left\|\begin{smallmatrix}10\\0\end{smallmatrix}\right\rangle$	$\|5\ \frac{-5}{2}\ 3;\ 3\ 0;\ h_1\ 0\ h_3;\ 2\rangle$	$h_1, h_3 = \pm1$
(8 states)	$\|5\ \frac{-5}{2}\ 3;\ 3\ 0;\ h_1\ 0\ h_3;\ 2\rangle$	$h_1, h_3 = \pm1$
$\left\|\begin{smallmatrix}11\\0\end{smallmatrix}\right\rangle$	$\|8\ 20\ 0;\ 0\ 0;\ 0\ 0\ 0;\ 3\rangle$	
(5 states)	$\|8\ 20\ 0;\ 3\ 3;\ h_1\ h_2\ 0;\ 3\rangle$	$h_1, h_2 = \pm1$
$\left\|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$	$\|8\ 20\ 0;\ 0\ 0;\ 0\ 0\ h_3;\ 2\rangle$	$h_3 = \pm2,0$
(15 states)	$\|8\ 20\ 0;\ 3\ 3;\ h_1\ h_2\ h_3;\ 2\rangle$	$h_1, h_2 = \pm1,\ h_3 = \pm2,0$
$\left\|\begin{smallmatrix}20\\0\end{smallmatrix}\right\rangle$	$\|12\ {-6}\ 0;\ 8\ 0;\ h_1\ 0\ 0;\ 3\rangle$	$h_1 = \pm2,0$
(10 states)	$\|12\ {-6}\ 0;\ 0\ 8;\ 0\ h_2\ 0;\ 3\rangle$ $\|12\ {-6}\ 0;\ 3\ 3;\ h_1\ h_2\ 0;\ 3\rangle$	$h_2 = \pm2,0$ $h_1, h_2 = \pm1$
$\left\|\begin{smallmatrix}20\\2\end{smallmatrix}\right\rangle$	$\|12\ {-6}\ 0;\ 8\ 0;\ h_1\ 0\ h_3;\ 2\rangle$	$h_1, h_3 = \pm2,0$
(30 states)	$\|12\ {-6}\ 0;\ 0\ 8;\ 0\ h_2\ h_3;\ 2\rangle$ $\|12\ {-6}\ 0;\ 3\ 3;\ h_1\ h_2\ h_3;\ 2\rangle$	$h_2, h_3 = \pm2,0$ $h_1, h_2 = \pm1, h_3 = \pm2,0$
$\left\|\begin{smallmatrix}21\\1\end{smallmatrix}\right\rangle$	$\|15\ \frac{75}{2}\ 3;\ 3\ 0;\ h_1\ 0\ h_3;\ 3\rangle$	$h_1, h_3 = \pm1$
(32 states)	$\|15\ \frac{75}{2}\ 3;\ 0\ 3;\ 0\ h_2\ h_3;\ 3\rangle$ $\|15\ \frac{75}{2}\ 3;\ 8\ 3;\ h_1\ h_2\ h_3;\ 3\rangle$ $\|15\ \frac{75}{2}\ 3;\ 3\ 8;\ h_1\ h_2\ h_3;\ 3\rangle$	$h_2, h_3 = \pm1$ $h_1 = \pm2,0,\ h_2,\ h_3 = \pm1$ $h_2 = \pm2,0,\ h_1,\ h_3 = \pm1$
$\left\|\begin{smallmatrix}21\\1\end{smallmatrix}\right\rangle$	$\|15\ \frac{75}{2}\ 3;\ 3\ 0;\ h_1\ 0\ h_3;\ 2\rangle$	$h_1, h_3 = \pm1$
(32 states)	$\|15\ \frac{75}{2}\ 3;\ 0\ 3;\ 0\ h_2\ h_3;\ 2\rangle$ $\|15\ \frac{75}{2}\ 3;\ 8\ 3;\ h_1\ h_2\ h_3;\ 2\rangle$ $\|15\ \frac{75}{2}\ 3;\ 3\ 8;\ h_1\ h_2\ h_3;\ 2\rangle$	$h_2, h_3 = \pm1$ $h_1 = \pm2,0, h_2, h_3 = \pm1$ $h_2 = \pm2,0, h_1, h_3 = \pm1$
$\left\|\begin{smallmatrix}21\\3\end{smallmatrix}\right\rangle$	$\|15\ \frac{75}{2}\ 3;\ 3\ 0;\ h_1\ 0\ h_3;\ 2\rangle$	$h_1 = \pm1,\ h_3 = \pm3, \pm1$
(64 states)	$\|15\ \frac{75}{2}\ 3;\ 0\ 3;\ 0\ h_2\ h_3;\ 2\rangle$ $\|15\ \frac{75}{2}\ 3;\ 8\ 3;\ h_1\ h_2\ h_3;\ 2\rangle$ $\|15\ \frac{75}{2}\ 3;\ 3\ 8;\ h_1\ h_2\ h_3;\ 2\rangle$	$h_2 = \pm1, h_3 = \pm3, \pm1$ $h_1 = \pm2, 0, h_2 = \pm1, h_3 = \pm3, \pm1$ $h_1 = \pm1, h_2 = \pm2,0, h_3 = \pm3, \pm1$
$\left\|\begin{smallmatrix}22\\0\end{smallmatrix}\right\rangle$	$\|20\ 110\ 0;\ 0\ 0;\ 0\ 0\ 0;\ 3\rangle$	
(14 states)	$\|20\ 110\ 0;\ 3\ 3;\ h_1\ h_2\ 0;\ 3\rangle$ $\|20\ 110\ 0;\ 8\ 8;\ h_1\ h_2\ 0;\ 3\rangle$	$h_1, h_2 = \pm1$ $h_1, h_2 = \pm2, 0$
$\left\|\begin{smallmatrix}22\\2\end{smallmatrix}\right\rangle$	$\|20\ 110\ 8;\ 0\ 0;\ 0\ 0\ h_3;\ 3\rangle$	$h_3 = \pm 2, 0$
(42 states)	$\|20\ 110\ 0;\ 3\ 3;\ h_1\ h_2\ h_3;\ 3\rangle$ $\|20\ 110\ 0;\ 8\ 8;\ h_1\ h_2\ h_3;\ 3\rangle$	$h_1, h_2 = \pm1, h_3 \pm 2, 0$ $h_1, h_2, h_3 = \pm2, 0$
$\left\|\begin{smallmatrix}30\\1\end{smallmatrix}\right\rangle$	$\|21\ \frac{-21}{2}\ 3;\ 15\ 0;\ h_1\ 0\ h_3;\ 3\rangle$	$h_1 = \pm3, \pm1, h_3 = \pm1$
(40 states)	$\|21\ \frac{-21}{2}\ 3;\ 0\ 15;\ 0\ h_2\ h_3;\ 3\rangle$ $\|21\ \frac{-21}{2}\ 3;\ 8\ \ 3;\ h_1\ h_2\ h_3;\ 3\rangle$ $\|21\ \frac{-21}{2}\ 3;\ 3\ \ 8;\ h_1\ h_2\ h_3;\ 3\rangle$	$h_2 = \pm 3, \pm1, h_3 = \pm1$ $h_1 = \pm 2, 0, h_2, h_3 = \pm1$ $h_2 = \pm 2, 0, h_1, h_3 = \pm1$
$\left\|\begin{smallmatrix}31\\0\end{smallmatrix}\right\rangle$	$\|24\ 60\ 0;\ 15\ 3;\ h_1\ h_2\ 0;\ 3\rangle$	$h_1 = \pm3, \pm1, h_2 = \pm1$
(35 states)	$\|24\ 60\ 0;\ 3\ 15;\ h_1\ h_2\ 0;\ 3\rangle$ $\|24\ 60\ 0;\ 8\ 0;\ h_1\ 0\ 0;\ 3\rangle$ $\|24\ 60\ 0;\ 0\ 8;\ 0\ h_2\ 0;\ 3\rangle$ $\|24\ 60\ 0;\ 8\ 8;\ h_1\ h_2\ 0;\ 3\rangle$ $\|24\ 60\ 0;\ 3\ 3;\ h_1\ h_2\ 0;\ 3\rangle$	$h_1 = \pm1, h_2 = \pm3, \pm1$ $h_1 = \pm2, 0$ $h_2 = \pm2, 0$ $h_1, h_2 = \pm2, 0$ $h_1, h_2 = \pm1$
$\left\|\begin{smallmatrix}31\\2\end{smallmatrix}\right\rangle$	$\|24\ 60\ 8;\ 15\ 3;\ h_1\ h_2\ h_3;\ 2\rangle$	$h_1 = \pm3, \pm1, h_2 = \pm1,$ $h_3 = \pm2, 0$
(105 states)	$\|24\ 60\ 8;\ 3\ 15;\ h_1\ h_2\ h_3;\ 2\rangle$ $\|24\ 60\ 8;\ 8\ 0;\ h_1\ 0\ h_3;\ 2\rangle$ $\|24\ 60\ 8;\ 0\ 8;\ 0\ h_2\ h_3;\ 2\rangle$ $\|24\ 60\ 8;\ 8\ 8;\ h_1\ h_2\ h_3;\ 2\rangle$ $\|24\ 60\ 8;\ 3\ 3;\ h_1\ h_2\ h_3;\ 2\rangle$	$h_1 = \pm1,\ h_2 = \pm3, \pm1, h_3 = \pm2, 0$ $h_1, h_3 = \pm2, 0$ $h_2, h_3 = \pm2, 0$ $h_1, h_2, h_3 = \pm2, 0$ $h_1, h_2 = \pm1, h_3 = \pm2, 0$
$\left\|\begin{smallmatrix}32\\1\end{smallmatrix}\right\rangle$	$\|29\ \frac{355}{2}\ 3;\ 15\ 8;\ h_1\ h_2\ h_3;\ 3\rangle$	$h_1 = \pm3, \pm1,$ $h_2 = \pm2, 0, h_3 = \pm1$
(80 states)	$\|29\ \frac{355}{2}\ 3;\ 8\ 15;\ h_1\ h_2\ h_3;\ 3\rangle$ $\|29\ \frac{355}{2}\ 3;\ 8\ 3;\ h_1\ h_2\ h_3;\ 3\rangle$ $\|29\ \frac{355}{2}\ 3;\ 3\ 8;\ h_1\ h_2\ h_3;\ 3\rangle$ $\|29\ \frac{355}{2}\ 3;\ 3\ 0;\ h_1\ h_2\ h_3;\ 3\rangle$ $\|29\ \frac{355}{2}\ 3;\ 0\ 3;\ h_1\ h_2\ h_3;\ 3\rangle$	$h_1 = \pm2, 0, h_2 = \pm3, \pm1, h_3 = \pm1$ $h_1 = \pm2, 0, h_2, h_3 = \pm1$ $h_2 = \pm2, 0, h_1, h_3 = \pm1,$ $h_1, h_3 = \pm1$ $h_2, h_3 = \pm1$

* For this representation, the Casimir operators of $\mathfrak{sp}(6)$ have the eigenvalues $(\rho_2, \rho_4, \rho_6) = (42, 6867, 59921)$.

4. Concluding remarks

Starting from the generalized Zhelobenko patterns (1) developed in [25, 26] for symplectic Lie algebras, we have derived an explicit branching rule for the irreducible representations of $\mathfrak{sp}(6)$ when reduced with respect to the maximal subalgebra $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$. Advantages of this formula over the generic patterns (1) reside in the possibility of classifying the multiplicity free reductions as well as solving the multiplicity problem for representations of the subalgebra intervening in the decomposition. In particular, using some of the inequalities of the (1) system, we are able to separate degeneracies in the decomposition by means of the scalar solutions of an equation. This procedure is used to construct bases of eigenstates for $\mathfrak{sp}(6)$ representations, but skipping the usually difficult computation of the missing label operator. Instead of this, we label repeated representations of the subalgebra using the solutions of equation (21), considered as a mere scalar equation. This alternative approach, mixing the combinatorial approach of (1) with the analytical ansatz to the missing label problem, is of practical use as it provides eigenstates for arbitrary representations $[k, l, m]$ directly from the decomposition formula (4). In contrast, for each *fixed* representation of $\mathfrak{sp}(6)$ the patterns (1) provide both the branching rule (with some additional computations) as well as the distinction of possible degeneracies, but for generic IRREPs the procedure is cumbersome because of its computational complications. The advantages, inconveniences, and the range of validity of this direct approach with respect to the general theory of S-functions developed in [21, 22] have been discussed, pointing out that for some specific problems, like the determination of the $\mathfrak{sp}(6)$ irreducible representations that contain a fixed IRREP of the subalgebra or finding the lowest dimensional $\mathfrak{sp}(6)$ representation, the decomposition of which exhibits a fixed multiplicity, the formula (4) and the pattern multiplicity criterion provide the answer in a reasonably simple way.

In principle, the same procedure developed here can be applied to obtain the precise branching rules for the reduction $\mathfrak{sp}(2N) \downarrow \mathfrak{sp}(2N-2) \times \mathfrak{sp}(2)$ with arbitrary $N > 3$. Although for these algebras the number of missing label operators is higher [33], it should be expected that the labelling problem can also be solved by means of scalars arising from a generalized equation (23).

The approach to the branching rules and labelling problem undertaken here could be useful to give an adequate solution to another intricacy in the representation theory: construction of matrix elements. For the reduction chain $\mathfrak{sp}(2N) \downarrow \mathfrak{sp}(2N-2) \times \mathfrak{sp}(2)$ there are still no general formulae for the matrix elements [11], and it is still an open problem whether our combinatorial approach, replacing the missing label operators by suitable scalars obtained from the equation (21), result in manipulable selection rules that enable to find the general expression for the matrix elements, as derived in [35] for the multiplicity free reduction in $N = 2$. The main difficulty in this aspect is to find suitable recurrence relations that can be solved for all $\mathfrak{sp}(2N)$ generators. This question is currently under close scrutiny, and we hope to find a satisfactory solution in the near future.

Acknowledgements

The author expresses his gratitude to the referees for suggesting several improvements of the manuscript as well as indicating the valuable references [21] and [22]. This work was partially supported by the research grant MTM2010-18556 of the Ministerio de Ciencia e Innovación (MICINN).

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$\mathfrak{sp}(6) \downarrow \mathfrak{sp}(4) \times \mathfrak{sp}(2)$ ŠAKOJIMOSI TAISYKLĖS IR TIKRINIŲ BŪSENŲ BAZĖS

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Santrauka

Pateikta išreikštinė formulė, aprašanti $\mathfrak{sp}(6)$ įvaizdžių šakojimąsi pagal redukcijos grandinėlę $\mathfrak{sp}(6) \downarrow \mathfrak{sp}(4) \times \mathfrak{sp}(2)$. Tai leidžia klasifikuoti redukcijas, neturinčias pasikartojimų, bei gauti pasikartojimų skaičių kiekvienam $\mathfrak{sp}(4) \times \mathfrak{sp}(2)$ įvaizdžiui. Metodas palyginamas su tuo, kas gaunama remiantis *S*-funkcijų teorija, nurodant išreikštinės formulės privalumus ir trūkumus. Šakojimosi taisyklė panaudota sudaryti ortogonalią $\mathfrak{sp}(6)$ tikrinių funkcijų bazę, kurioje išsigimimai sutvarkomi taikant skaliarą vietoje įprastinio trūkstamos žymos operatoriaus.