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MAJORANA SPINOR FROM THE POINT OF VIEW OF GEOMETRIC ALGEBRA

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Majorana spinors are constructed in terms of the multivectors of relativistic $Cl_{1,3}$ algebra. Such spinors are found to be multiplied by primitive idempotents which drastically change spinor properties. Running electronic waves are used to solve the real Dirac–Majorana equation transformed to $Cl_{1,3}$ algebra. From the analysis of the solution it is concluded that free Majorana particles do not exist, because relativistic $Cl_{1,3}$ algebra requires the massive Majorana particle to move with light velocity.

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1. Introduction

Solutions of the Dirac equation with the electron charge equated to zero, $e = 0$, are called the Majorana spinors [1]. In the matrix notation the Majorana spinors are real-valued column-vectors which come from the assumption that all Dirac- $\hat{\gamma}_i$'s are real. Thus, the respective Majorana fields represent the particles that are their own antiparticles. Since the particles and antiparticles coincide, from this it follows that the Majorana particle is a neutral spin-$\frac{1}{2}$ particle (fermion) carrying no charge. The concept of Majorana spinor, especially in the condensed matter physics, is frequently generalized assuming that the spinor is real but not necessarily related to the Dirac equation. In terms of the Hilbert space the properties of Majorana spinors were discussed most thoroughly in articles [2, 3], where the earlier literature is cited.

Nonetheless, there is no final consensus about the status of Dirac–Majorana equation (also called the Majorana equation) in physics. For example, recently the authors [4] came to the conclusion that "the physical meaning of Majorana's equation is very dubious. Therefore, it seems to us that this equation cannot give the equation of motion of the neutral WIMPs (weakly interacting massive particles), the hypothesized constitutive elements of the Dark Matter". On the other hand, in the condensed matter physics the Majorana equation and the respective particles are addressed to explain the physics of a topological superconductor (superconductor film carrying a vortex in contact with a 3D topological insulator), where the Majorana fermion bound to a vortex makes a qubit which is expected to have an unusually long coherence time and therefore may find application in the quantum computer development. The present-day theoretical as well as the experimental status on Majorana fermions may be found, for example, in review articles [5] and [6].

The Dirac–Majorana (DM) equation

$$\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\psi^{M} - m\psi^{M} = 0, \tag{1}$$

where m is the mass, $\mathrm{i}\hat{\gamma}^{\mu}$ are real 4×4 Majorana matrices [7] and ψ^M is a real spinor, as a rule, is analyzed in terms of the complex Hilbert space [2, 3]. In the present paper the DM equation is formulated and solved in terms of real geometric algebra (GA) [8–11], by mathematicians also called the Clifford algebra, where the quantum mechanics is formulated in graded vector spaces that accommodate non-commuting multivectors. Two types of geometrical algebras,

$Cl_{1,3}$ and $Cl_{3,1}$, having, respectively, (+,–,–,–) and (+,+,+,–) signatures are used to represent the Dirac equation [11]. The first, usually addressed by physicists, may be represented either by 2×2 quaternionic irreducible matrices or by complex 4×4 matrices which are usually used in solving the problems. The second, $Cl_{3,1}$ algebra, is isomorphic to real 4×4 matrices. This paper is limited to the analysis of DM equation by $Cl_{1,3}$ algebra only. In the conclusions an attempt to solve the DM equation by $Cl_{3,1}$ algebra will also be mentioned.

2. Spinors in geometric algebra

Two approaches based on either left minimal ideals or even subalgebras $Cl^+_{p,q} \in Cl_{p,q}$, expressed in some basis are addressed to describe the spinors in GA [12, 13]. In practice, the second approach, where one of the basis vectors is selected as a quantization axis, is preferred in physics and used more frequently. Of all Clifford groups that describe the internal structure of GA, the most important from the physics point of view is the spin group Spin(p,q), also called the spinor group, which is directly connected with $Cl_{p,q}$ algebra. Thus the GA itself is capable to construct the half-spin representations of Spin(p,q) groups [12, 14]. Both algebras, $Cl_{1,3}$ and $Cl_{3,1}$, are described by the same spin group

$$\mathrm{Spin}_{3,1} = \mathrm{Spin}_{1,3} \cong \mathrm{SL}(2, \mathbb{C}), \tag{2}$$

which is isomorphic to a special linear (SL) group represented by 2×2 complex matrices. The number of elements (infinitesimal generators) in SL (2, $\mathbb{C}$) is $\frac{1}{2}n(n-1)$, where $n = p + q$ [13]. From this it follows that all properties of the $\mathrm{Spin}_{1,3}$ group may be represented by six complex 2×2 matrices.

The spin (or spinor) group may also be constructed from GA basis vectors $\mathbf{e}_i$ that obey the anti-commutation relation $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\delta_{ij}$, where δ_{ij} is the standard Kronecker symbol. The number of basis vectors $\mathbf{e}_i$ is equal to $n = p + q$. The spinor is constructed from bivectors, grade-2 GA elements $\mathcal{B} = \mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i$, where here and in the following it is assumed that the basis is orthonormal so that the wedge product $\wedge$ may be replaced by a geometric product. The number of bivectors in $Cl_{p,q}$ may be expressed through the binomial $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{1}{2}n(n-1)$, where $n = p + q$, which coincides with the order (number of elements) of the SL(2, $\mathbb{C}$) group. The elements of the spin group may be represented by the exponential of bivectors $\mathcal{B}$, in the form $\mathrm{e}^{\pm\mathcal{B}} \in \mathrm{Spin}_{p,q}$. Then a general group element will be a geometric product of exponentials. Since $\mathrm{e}^{\pm\mathcal{B}}$ in the expanded form consists of a scalar and a bivector, the product of exponentials will generate an even graded multivector. Thus, the general spinor in $Cl_{1,3}$ has the following form:

$$\psi = a^0 + a^1\mathbf{e}_4\mathbf{e}_3 + a^2\mathbf{e}_2\mathbf{e}_4 + a^3\mathbf{e}_3\mathbf{e}_2$$
$$+ b^0\mathbf{e}_4\mathbf{e}_1 + b^1\mathbf{e}_3\mathbf{e}_1 + b^2\mathbf{e}1\mathbf{e}_2 + b^3 I. \tag{3}$$

Here a^i and b^i are real coefficients and I is the pseudoscalar, $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$.

The basis vectors $\mathbf{e}_i$ will be replaced by non-hatted gammas γ_μ, where $\mu = 0,1,2,3$ and $\gamma_0^2 = 1$. The remaining gammas satisfy $\gamma_\mu^2 = -1$. Following the book [11] (for details see Appendix 5), the bivectors will be denoted by the sigmas: $\boldsymbol{\sigma}_1 = \gamma_1\gamma_0$, $\boldsymbol{\sigma}_2 = \gamma_2\gamma_0$, $\boldsymbol{\sigma}_3 = \gamma_3\gamma_0$, $I\boldsymbol{\sigma}_1 = \gamma_3\gamma_2$, $I\boldsymbol{\sigma}_2 = \gamma_1\gamma_3$, $I\boldsymbol{\sigma}_3 = \gamma_2\gamma_1$. The bivectors $\boldsymbol{\sigma}_i$ are space-like, $\boldsymbol{\sigma}_i^2 = 1$, and the bivectors $I\boldsymbol{\sigma}_i$ are time-like, $(I\boldsymbol{\sigma}_i)^2 = -1$.

There exists a bilateral correspondence and respective replacement rules among the GA spinors ϕ and the Hilbert space ket-spinors $|\phi\rangle$ represented by columns. For the Pauli spinor the replacement rule between the Hilbert space and GA is [11]

$$|\phi\rangle = \begin{bmatrix} a^0 + \mathrm{i}a^3 \\ -a^2 + \mathrm{i}a^1 \end{bmatrix} \Leftrightarrow \phi$$
$$= a^0 + a^1 I\boldsymbol{\sigma}_1 + a^2 I\boldsymbol{\sigma}_2 + a^3 I\boldsymbol{\sigma}_3 \equiv a^0 + a^k I\boldsymbol{\sigma}_k, \tag{4}$$

which shows how the scalar coefficients a^i in the 4-component spinor of $Cl_{3,0}$ are related with the complex coefficients of the Hilbert space spinor. In GA, the normalization condition for the spinor is $\phi\tilde{\phi} = 1$, where the tilde indicates the reversion involution the meaning of which is seen from the example $\widetilde{\mathbf{e}_i\mathbf{e}_j} = \mathbf{e}_j\mathbf{e}_i$. The standard 4-component Dirac spinor $|\psi\rangle$ is obtained by stacking two Pauli spinors (4). In $Cl_{1,3}$ algebra, as Eq. (3) shows, the GA spinor ψ is a sum of a scalar, six bivectors, and a pseudoscalar. Then it can be shown that for the relativistic $Cl_{1,3}$ algebra the replacement rule reads

$$|\psi\rangle = \begin{bmatrix} a^0 + \mathrm{i}a^3 \\ -a^2 + \mathrm{i}a^1 \\ b^0 + \mathrm{i}b^3 \\ -b^2 + \mathrm{i}b^1 \end{bmatrix} \Leftrightarrow \psi$$
$$= a^0 + a^k I\boldsymbol{\sigma}_k + (b^0 + b^k I\boldsymbol{\sigma}_k)\boldsymbol{\sigma}_3. \tag{5}$$

Map (5) may be written more concisely in the form of two Pauli spinors χ and ω:

$$|\psi\rangle = \begin{bmatrix} |\chi\rangle \\ |\omega\rangle \end{bmatrix} \Leftrightarrow \psi = \chi + \omega\boldsymbol{\sigma}_3. \tag{6}$$

The action of Dirac gamma matrices or their combinations onto column-spinors has an equivalent expression in GA. Rules (5) and (6) induce the following basic replacement rules for transition from the gamma matrix action onto the column-spinor to a respective GA multivector expression

$$\begin{aligned}
&\hat{\gamma}_\mu|\psi\rangle \Leftrightarrow \gamma_\mu\psi\gamma_0,\\
&\mathrm{i}|\psi\rangle \Leftrightarrow \psi I\boldsymbol{\sigma}_3,\\
&|\psi^*\rangle \Leftrightarrow -\gamma_2\psi\gamma_2,\\
&\langle\psi|\phi\rangle \Leftrightarrow \langle\gamma_0\tilde{\psi}\gamma_0\phi\rangle - \langle\gamma_0\tilde{\psi}\gamma_0\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3,
\end{aligned} \tag{7}$$

where $\mathrm{i}=\sqrt{-1}$ and an asterisk means the complex conjugation. The bracket $\langle\ldots\rangle$ that appears in the right-hand side means that only the scalar part is to be taken from the resulting multivector inside the bracket. The square of the spinor norm, as follows from the last line and spinor (5), is

$$|\psi|^2=\langle\psi\gamma_0\tilde{\psi}\gamma_0\rangle=a_0^2+a_1^2+a_2^2+a_3^2+b_0^2+b_1^2+b_2^2+b_3^2. \tag{8}$$

The above rules and in particular the last expression in (7) allow calculating the replacement rule for the matrix element of the operator $\hat{O}$,

$$\langle\psi|\hat{O}|\psi\rangle \Leftrightarrow \langle\gamma_0\tilde{\psi}\gamma_0 O^C(\psi)\rangle - \langle\gamma_0\tilde{\psi}\gamma_0 O^C(\psi)I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3, \tag{9}$$

where O^C is the operator $\hat{O}$ transformed to GA, $\hat{O}|\psi\rangle \Leftrightarrow O^C(\psi)$.

It is important to stress that in GA the spinor ψ in principle is free of matrix representation, i. e. it is coordinate-free. The appearance of basis vectors or bivectors in the Dirac and Majorana equations in the GA form is related with the necessity to declare the time axis and the spin quantization axis, which in our case are represented by the basis vector γ_0 and the oriented plane $I\boldsymbol{\sigma}_3$. The appearance of γ_2 in (7) is not related with any physical axis or oriented plane. Since the operation of complex conjugation does not belong to GA involutions, the rule $|\psi^*\rangle \Leftrightarrow -\gamma_2\psi\gamma_2$ only shows a formal equivalence between the signs at scalar coefficients in spinors (3) and (5).

3. Majorana spinors in GA

In the standard or Pauli–Dirac representation the gamma 4×4 matrices are

$$\hat{\gamma}_0=\begin{bmatrix}\hat{1}_2 & 0\\ 0 & \hat{1}_2\end{bmatrix},\quad \hat{\gamma}_k=\begin{bmatrix}0 & -\hat{\boldsymbol{\sigma}}_k\\ \hat{\boldsymbol{\sigma}}_k & 0\end{bmatrix},\ k=1,2,3, \tag{10}$$

where $\hat{1}_2$ is the 2×2 unit matrix and $\hat{\boldsymbol{\sigma}}_k$ are Pauli matrices. To have projective operators in the Hilbert space an additional fifth gamma matrix, $\hat{\gamma}_5=-\mathrm{i}\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$, is introduced. To avoid misinterpretations in applying the replacement rules the transition to and back will be referenced with respect to the standard Pauli–Dirac representation (10).

The Dirac gammas in the Majorana representation can be found with the unitary matrix [15]

$$\hat{U}^M=\frac{1}{\sqrt{2}}\begin{bmatrix}\hat{1}_2 & \hat{\boldsymbol{\sigma}}_2\\ \hat{\boldsymbol{\sigma}}_2 & \hat{1}_2\end{bmatrix},\ (\hat{U}^M)^{-1}=(\hat{U}^M)^\dagger=\hat{U}^M, \tag{11}$$

which satisfies $\hat{U}^M(\hat{U}^M)^{-1}=(\hat{U}^M)^2=\hat{1}$. Then, after unitary transformation the gamma matrices (10) become purely imaginary [7]:

$$\hat{\hat{\gamma}}_0=\begin{bmatrix}0 & \hat{\boldsymbol{\sigma}}_2\\ \hat{\boldsymbol{\sigma}}_2 & 0\end{bmatrix},\quad \hat{\hat{\gamma}}_1=\begin{bmatrix}-\mathrm{i}\hat{\boldsymbol{\sigma}}_3 & 0\\ 0 & -\mathrm{i}\hat{\boldsymbol{\sigma}}_3\end{bmatrix}, \tag{12}$$

$$\hat{\hat{\gamma}}_2=\begin{bmatrix}0 & \hat{\boldsymbol{\sigma}}_2\\ -\hat{\boldsymbol{\sigma}}_2 & 0\end{bmatrix},\quad \hat{\hat{\gamma}}_3=\begin{bmatrix}\mathrm{i}\hat{\boldsymbol{\sigma}}_1 & 0\\ 0 & \mathrm{i}\hat{\boldsymbol{\sigma}}_1\end{bmatrix}. \tag{13}$$

We shall assume that the Majorana spinor (*pro tempore* complex) has the form

$$|\boldsymbol{\psi}^M\rangle=\begin{bmatrix}a^0+\mathrm{i}b^0\\ -a^2-\mathrm{i}b^2\\ a^1+\mathrm{i}b^1\\ -a^3-\mathrm{i}b^3\end{bmatrix}
=\frac{1}{2}\begin{bmatrix}|\chi+\chi^*\rangle+\mathrm{i}\,|\omega+\omega^*\rangle\\ \hat{\boldsymbol{\sigma}}_2\,|\chi-\chi^*\rangle+\mathrm{i}\,\hat{\boldsymbol{\sigma}}_2\,|\omega-\omega^*\rangle\end{bmatrix}, \tag{14}$$

where a^i and b^i are scalar coefficients in the 2-component spinors $|\chi\rangle\rightarrow\chi=a^0+a^kI\boldsymbol{\sigma}_k$ and $|\omega\rangle\rightarrow\omega=b^0+b^kI\boldsymbol{\sigma}_k$. Then, the spinor in the standard Dirac representation becomes

$$\begin{aligned}
&|\psi\rangle=(\hat{U}^M)^\dagger\,|\boldsymbol{\psi}^M\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}|\chi\rangle+\mathrm{i}\,|\omega\rangle\\ \boldsymbol{\sigma}_2\,(|\chi\rangle-\mathrm{i}\,|\omega\rangle)^*\end{bmatrix}\Leftrightarrow\\
&\psi=\frac{1}{\sqrt{2}}(\chi+\omega I\boldsymbol{\sigma}_3)+\frac{1}{\sqrt{2}}\boldsymbol{\sigma}_2\boldsymbol{\sigma}_2(\chi-\omega I\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\boldsymbol{\sigma}_3\\
&=\frac{1}{\sqrt{2}}\chi(1+\boldsymbol{\sigma}_2)+\frac{1}{\sqrt{2}}\omega I\boldsymbol{\sigma}_3(1-\boldsymbol{\sigma}_2).
\end{aligned} \tag{15}$$

The dagger at $\hat{U}^M$ suggests that the spinor was transformed to the standard representation and then the replacement rules (6) and (7) were applied [16]. Below the properties of ψ in (15) that represents the Majorana spinor in GA are discussed.

1. In (15) there appear two projectors, $P_+ = \frac{1}{2}(1+\boldsymbol{\sigma}_2)$ and $P_- = \frac{1}{2}(1-\boldsymbol{\sigma}_2)$, which divide the spinor into two complimentary parts that can be swapped if ψ is multiplied by the bivector $I\boldsymbol{\sigma}_3$, i. e. $\psi I\boldsymbol{\sigma}_3 = \frac{1}{\sqrt{2}}\chi I\boldsymbol{\sigma}_3(1-\boldsymbol{\sigma}_2) - \frac{1}{\sqrt{2}}\omega(1+\boldsymbol{\sigma}_2)$. If transformed to the standard Pauli–Dirac representation using replacement rules (7), these projectors become

$$\hat{P}_{\pm}^{D} = \frac{1}{2}\begin{bmatrix} \hat{1}_2 & \pm\hat{\boldsymbol{\sigma}}_2 \\ \pm\hat{\boldsymbol{\sigma}}_2 & \hat{1}_2 \end{bmatrix}, \qquad (16)$$

which after transformation to the Majorana representation assume a block-diagonal form that is typical of projectors:

$$\hat{P}_{+}^{M} = \begin{bmatrix} \hat{1}_2 & \hat{0} \\ \hat{0} & \hat{0} \end{bmatrix},\quad \hat{P}_{-}^{M} = \begin{bmatrix} \hat{0} & \hat{0} \\ \hat{0} & \hat{1}_2 \end{bmatrix}. \qquad (17)$$

The above matrices are similar to the helicity projection matrices $(1 \pm \hat{\gamma}_5)/2$ in the standard Dirac representation.

2. In (15), the Majorana spinor becomes real when $\omega = 0$, i. e. when the second term vanishes,

$$|\psi^M\rangle = [a^0, -a^2, a^1, -a^3]^{\mathrm{T}} \Leftrightarrow \psi^M = \frac{1}{\sqrt{2}}\chi(1+\boldsymbol{\sigma}_2)$$
$$= \frac{1}{\sqrt{2}}(a^0 + a^1 I\boldsymbol{\sigma}_1 + a^2 I\boldsymbol{\sigma}_2 + a^3 I\boldsymbol{\sigma}_3)(1+\boldsymbol{\sigma}_2). \qquad (18)$$

As the last line shows, χ may be interpreted as the $Cl_{3,0}$ algebra spinor and therefore can be reduced to a product of 3D Euclidean space rotors. Thus Eq. (18) shows that all real Majorana spinors are described by all possible sets of three dimensional rotors R. From this we conclude that the real Majorana spinor may be replaced by the product of two 3D rotors parameterized by angles, θ and φ,

$$\chi = \mathrm{R}(\varphi)\mathrm{R}(\theta) = \mathrm{e}^{-\varphi I\boldsymbol{\sigma}_3/2}\mathrm{e}^{-\theta I\boldsymbol{\sigma}_2/2}$$
$$= \cos\frac{\theta}{2}\cos\frac{\varphi}{2} + I\boldsymbol{\sigma}_1\sin\frac{\theta}{2}\sin\frac{\varphi}{2}$$
$$- I\boldsymbol{\sigma}_2\sin\frac{\theta}{2}\cos\frac{\varphi}{2} - I\boldsymbol{\sigma}_3\cos\frac{\theta}{2}\sin\frac{\varphi}{2}, \qquad (19)$$

and then, as (18) requires, multiplied by a respective projector. Spinor (19) satisfies $\chi\widetilde{\chi} = 1$. From this we conclude that the spinor χ is nothing else but a 3D Euclidean space rotor the coefficients a_i of which can be parameterized by two rotation angles in the Euclidean space: $a^0 = \cos\frac{\theta}{2}\cos\frac{\varphi}{2}$, $a^1 = \sin\frac{\theta}{2}\sin\frac{\varphi}{2}$, $a^2 = -\sin\frac{\theta}{2}\cos\frac{\varphi}{2}$, $a^3 = -\cos\frac{\theta}{2}\sin\frac{\varphi}{2}$. Thus, all possible states of χ are isomorphic to the Pauli spinor-rotor of $Cl_{3,0}$ algebra and, as known, represent points on the Bloch sphere where every point is related with a quantum state of the two-level system which can be reached by the spinor-rotor χ. In particular, the two basis states on the Bloch sphere, which are equivalent to up $|\uparrow\rangle$ and down $|\downarrow\rangle$ spin states, can be described by rules $|\uparrow\rangle \to 1$ and $|\downarrow\rangle \to -I\boldsymbol{\sigma}_2$. They correspond, respectively, to angles $(\varphi, \theta) = (0, 0)$ and $(\varphi, \theta) = (0, \pi)$ in (19). The other states on the Bloch sphere can be written as linear combinations, $|\psi = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, where the complex constants α and β determine all possible states. They are totally equivalent to the GA spinor-rotor (19).

3. As mentioned, the Majorana spinor (18) is real and consists of the product of the Euclidean space rotor χ and the projector (idempotent). On the other hand, the standard Pauli–Dirac spinor consists of the product of the Euclidean space rotor and the relativistic boost [11]. In GA they are represented by exponents that generate trigonometric functions in the case of rotations in the physical space and the hyperbolic functions in the case of boost, or acceleration of the system. Anyway, in the Majorana spinor (18) the idempotent cannot be reduced to hyperbolic functions, i. e. boosts of the system. This can be seen from the expression for the standard boost $\mathrm{e}^{\vartheta\boldsymbol{\sigma}_2/2} = \cosh(\vartheta/2) + \boldsymbol{\sigma}_2\sinh(\vartheta/2)$, which cannot be transformed to a characteristic idempotent form $(1\pm\boldsymbol{\sigma}_2)/2$ by varying the boost angle ϑ.

4. If $\chi = 0$, the spinor is imaginary, and ψ becomes

$$\psi_i^M = 2^{-1/2} I\boldsymbol{\sigma}_3\,\omega(1 - \boldsymbol{\sigma}_2), \qquad (20)$$

where ω represents the Pauli spinor. The factor $(1 - \boldsymbol{\sigma}_2)/2$ is the idempotent which projects onto the complimentary space. Apart from the phase factor $I\boldsymbol{\sigma}_3 = \mathrm{e}^{\pi I\boldsymbol{\sigma}_3/2}$, this spinor is similar to (18). In the following the real part of the spinor, i. e. Eq. (18), will be used.

4. Dirac–Majorana equation

The Majorana equation is

$$(\mathrm{i}\,\hat{\bar{\gamma}}^{\mu}\partial_{\mu} - m)\,|\,\psi\rangle = 0, \qquad (21)$$

where $|\psi\rangle$ is the Majorana spinor and $\hat{\bar{\gamma}}^{\mu}$ are purely imaginary (indicated by overbar) gamma matrices (12) and (13), therefore, $\mathrm{i}\hat{\bar{\gamma}}^{\mu} = -\hat{\gamma}^{\mu}$ are real matrices. The Majorana spinors are 4-dimensional real column-vectors. So the real Majorana equation can be rewritten as

$$(\hat{\gamma}^{\mu}\partial_{\mu} + m)\,|\,\psi\rangle = 0. \qquad (22)$$

If the complex conjugation is taken, we have

$$(\hat{\gamma}^{\mu}\partial_{\mu} + m) \mid \psi^{*}\rangle = 0. \quad (23)$$

From this it follows that the Majorana spinor is a real quantity: $|\psi^{*} = |\psi\rangle$.
If a differential vector operator (nabla) is introduced,

$$\nabla = \gamma_0\partial_t - \gamma_1\partial_x - \gamma_2\partial_y - \gamma_3\partial_z, \quad (24)$$

then the Majorana equation (22) in a coordinate-free GA notation will read

$$\nabla\psi I\boldsymbol{\sigma}_3 + m\psi\gamma_0 = 0. \quad (25)$$

The solution in the form of plane running electronic waves will be sought,

$$\psi = \psi_0 e^{\pm I\sigma_3 p\cdot x} = 0, \quad (26)$$

where different signs in the exponent correspond to forward and backward waves, and ψ_0 is a constant spinor (18). The inner, or dot, product of the 4-momentum $p = p_0\gamma_0 + p_1\gamma_1 + p_2\gamma_2 + p_3\gamma_3$ and the 4-coordinate $x = t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3$ gives the scalar phase:

$$p\cdot x = p_0x_0 - p_1x_1 - p_2x_2 - p_3x_3. \quad (27)$$

The action of the 4-dimensional nabla upon the running wave exponential gives

$$\nabla\psi = \nabla(\psi_0 e^{\pm I\sigma_3 p\cdot x}) = \pm p\psi I\boldsymbol{\sigma}_3. \quad (28)$$

Thus, under the running wave ansatz (26) the Majorana equation (25) becomes algebraic,

$$\mp p\psi_0 - m\psi_0\gamma_0 = 0, \quad (29)$$

or, after insertion of the Majorana real spinor from Eq. (18), $\psi_0 = \psi^M = \frac{1}{\sqrt{2}}\chi(1+\boldsymbol{\sigma}_2)$, it becomes

$$\pm p\chi(1+\boldsymbol{\sigma}_2) = -m\chi(1+\boldsymbol{\sigma}_2)\gamma_0. \quad (30)$$

The idempotents on the left and right side of (30) cannot be cancelled out, since only the projections rather than the projected expressions before idempotents are equal. Also note that $\psi^M\widetilde{\psi}^M = 0$, so that the spinor is non-normalizable. However, the 3D spinor χ that belongs to the $R^{3,0}$ subspace is well defined and satisfies the normalization, $\chi\widetilde{\chi} = 1$. Therefore, Eq. (29) can be rewritten (after right multiplication by $\widetilde{\chi}$) as

$$\pm\widetilde{\chi}p\chi(1+\boldsymbol{\sigma}_2) = -m\chi(1+\boldsymbol{\sigma}_2)\gamma_0. \quad (31)$$

Now the space-time splitting will be performed by multiplying both sides of (31) by γ_0 from left and right. Noting that χ and γ_0 commute, and the right splitting of the 4-momentum gives $p_0\gamma_0 = E_0 + \mathbf{p}$, and the left splitting gives $\gamma_0 p = E_0 - \mathbf{p}$, where $\mathbf{p} = p_1\boldsymbol{\sigma}_1 + p_2\boldsymbol{\sigma}_2 + p_3\boldsymbol{\sigma}_3$ is the classical momentum and $E_0 \equiv p_0$ is the full energy of the system, one finds two equations:

$$\pm(E_0 + \widetilde{\chi}\mathbf{p}\chi(1-\boldsymbol{\sigma}_2) = -m(1+\boldsymbol{\sigma}_2), \quad (32)$$

$$\pm(E_0 - \widetilde{\chi}\mathbf{p}\chi(1+\boldsymbol{\sigma}_2) = -m(1-\boldsymbol{\sigma}_2). \quad (33)$$

It should be noted that different, complimentary idempotents appear on the left- and right-hand sides of these equations. The sum of (32) and (33) gives the relation between the energy and 3D linear momentum,

$$\pm(E_0 - \widetilde{\chi}\mathbf{p}\chi\boldsymbol{\sigma}_2) = -m, \quad (34)$$

from which the classical momentum $\mathbf{p}$ can be expressed as

$$\mathbf{p} = (E_0 \pm m)(\chi\boldsymbol{\sigma}_2\widetilde{\chi}), \quad (35)$$

where the upper/lower signs correspond to respective signs in the running wave expression (26). The difference between (32) and (33) gives the same result (34). The following conclusions can be drawn from the above equations.

1. Since

$$\chi\boldsymbol{\sigma}_2\widetilde{\chi} = \sin\varphi\boldsymbol{\sigma}_1 + \cos\varphi\boldsymbol{\sigma}_2, \quad (36)$$

it follows that the linear momentum $\mathbf{p}$ lies in the $\boldsymbol{\sigma}_1$–$\boldsymbol{\sigma}_2$ plane and is characterized by the angle φ and the magnitude $(E_0 \pm m)$:

$$\mathbf{p} = (E_0 \pm m)(\sin\varphi\boldsymbol{\sigma}_1 + \cos\varphi\boldsymbol{\sigma}_2). \quad (37)$$

The polar angle θ in (36) is absent because the rotation of $\boldsymbol{\sigma}_2$ by χ was done in the $I\boldsymbol{\sigma}_2$ plane which is perpendicular to the $\boldsymbol{\sigma}_2$ axis of $Cl_{3,0}$. If rotations of remaining basis vectors are performed, there appears the polar angle θ:

$$\chi\boldsymbol{\sigma}_1\widetilde{\chi} = \cos\theta\cos\varphi\boldsymbol{\sigma}_1 - \cos\theta\sin\varphi\boldsymbol{\sigma}_2 + \sin\theta\boldsymbol{\sigma}_3,$$

$$\chi\boldsymbol{\sigma}_3\widetilde{\chi} = -\sin\theta\cos\varphi\boldsymbol{\sigma}_1 - \sin\theta\sin\varphi\boldsymbol{\sigma}_2 + \cos\theta\boldsymbol{\sigma}_3. \quad (38)$$

2. At $\mathbf{p} = 0$ we have the energy gap $E_0 = \pm m$. The negative energy, as we shall see below, should be rejected.

3. If $\varphi = 0$, from (36) we have $\mathbf{p} = (E_0 \pm m)\boldsymbol{\sigma}_2$, the magnitude of which gives the graphene-like spectrum $E_0 = |\mathbf{p}| \mp m$. The minus sign at the mass term gives the zero total energy for a particle at the nonzero momentum and, therefore, should be rejected as unphysical. Expression (35) shows that the character of the spectrum remains the same in other directions determined by the angle φ in (36). From all of this it follows that the spectrum is described by a shifted upper half of the graphene cone,

$$E_0 = |\mathbf{p}| + m \text{ spectrum of DM equation in } Cl_{1,3}, \quad (39)$$

as shown in Fig. 1. The momentum lies in the plane $\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$. In the dimensional units the spectrum is $E_0 = c|\mathbf{p}| + mc^2$. Since at the energies $E_0 > m$ it gives the velocity for a Majorana particle equal to light velocity, $|\mathbf{v}| = c$, the spectrum should be rejected as unphysical.

A few words are appropriate here to say about the solution of the DM equation within the framework of $Cl_{3,1}$ algebra [17]. This algebra may be represented by real 4×4 matrices, so at the first sight it may appear that $Cl_{3,1}$ algebra is an ideal coordinate-free tool to analyze the point at issue. However, here one encounters the problem of compatibility between the replacement rules for real quantum mechanics and $Cl_{3,1}$ algebra with real spinors. It appears that such rules can be constructed only for its even subalgebra $Cl^+_{3,1} \in Cl_{3,1}$ but not for a full $Cl_{3,1}$ algebra. As a result, the real Dirac–Majorana equation cannot be written in terms of $Cl_{3,1}$, although the real spinors do exist.

In summary, the Majorana real spinors were calculated and then applied to the DM equation using the relativistic geometric algebra $Cl_{1,3}$. It was found that the real spinors in $Cl_{1,3}$ are equal to the $Cl_{3,0}$ spinors multiplied by an idempotent element. As a result, the unphysical solutions of the DM equation are obtained for running waves, implying that the Majorana particle does not exist. An attempt to solve the DM equation with running waves within the framework of Hilbert space formulation also gives an unphysical spectrum [17]. All solutions considered here correspond to infinite boundary conditions. The solutions at finite boundary conditions, without using the running to infinity waves, were not investigated in this paper. However, from the study of topological insulators we know that the bounded solutions of the DM equation may become physical in such cases [5, 6].

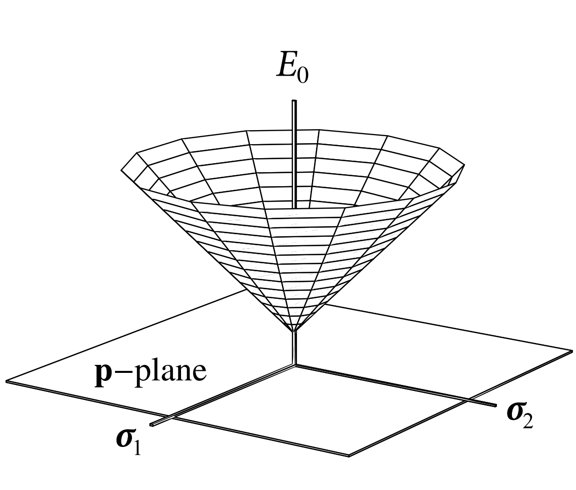

Fig. 1. Single-cone described by dispersion $E_0 = |\mathbf{p}| + m$. The energy gap between the cone tip and wave vector plane is equal to mass m.

Appendix

In GA, the orthogonal basis vectors are $\mathbf{e}_i$, the bivectors are $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$, etc. As usual, the geometric product, for example $\mathbf{e}_i\mathbf{e}_j$, has no special symbol. In relativistic $Cl_{1,3}$ algebra, however, a different and more convenient notation is preferred, which reconciles multivector notations of classical $Cl_{3,0}$ and relativistic $Cl_{1,3}$ algebras. We shall remind that $Cl_{3,0}$ is an even subalgebra of $Cl_{1,3}$. In the present paper the notation of [11] is used. In this notation the basis vectors $\mathbf{e}_i$ are replaced by Dirac gammas symbols γ_μ. Below the notation used is summarized.

Vectors: $\gamma_0 = \mathbf{e}_1$, $\gamma_1 = \mathbf{e}_2$, $\gamma_2 = \mathbf{e}_3$, $\gamma_3 = \mathbf{e}_4$;

$\gamma_0^2 = 1$ and $\gamma_k^2 = -1$ when $k = 1,2,3$.

Bivectors: $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$, $\boldsymbol{\sigma}_k^2 = 1$;

$I\boldsymbol{\sigma}_k \equiv I\boldsymbol{\sigma}_k$, $(I\boldsymbol{\sigma}_k)^2 = -1$;

$I\boldsymbol{\sigma}_1 = \gamma_3\gamma_2$, $I\boldsymbol{\sigma}_2 = \gamma_1\gamma_3$, $I\boldsymbol{\sigma}_3 = \gamma_2\gamma_1$.

Trivectors: $I\gamma_\mu \equiv I\gamma_\mu$, $\mu = 0,1,2,3$;

$I\gamma_0 = -\gamma_1\gamma_2\gamma_3$, $(I\gamma_0)^2 = 1$;

$I\gamma_1 = -\gamma_0\gamma_2\gamma_3$, $(I\gamma_1)^2 = -1$;

$I\gamma_2 = \gamma_0\gamma_1\gamma_3$, $(I\gamma_2)^2 = -1$;

$I\gamma_3 = -\gamma_0\gamma_1\gamma_2$, $(I\gamma_3)^2 = -1$.

Pseudoscalar:

$I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 = \gamma_0\gamma_1\gamma_2\gamma_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$;

$I^2 = -1$, $I\widetilde{I} = -1$.

After the spacetime splitting (multiplication by γ_0), the considered notation allows treating the bivector symbols $\{\sigma_1\sigma_2\sigma_3\}$ of $Cl_{1,3}$ as vectors in $Cl_{3,0}$ and the symbols $\{I\sigma_1 I\sigma_2 I\sigma_3\}$ as bivectors in $Cl_{3,0}$. Thus, after the splitting the basis elements belong to $Cl_{3,0}$ and become observer dependent. This is why they are written in bold, $\{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\}$ and $\{\boldsymbol{I\sigma}_1\boldsymbol{I\sigma}_2\boldsymbol{I\sigma}_3\}$, i. e. as vectors and bivectors of $Cl_{3,0}$. The scalar and the pseudoscalar are shared by both algebras.

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MAJORANOS SPINORIAI GEOMETRINĖS ALGEBROS POŽIŪRIU

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Santrauka

Pritaikius geometrinės algebros multivektorius buvo sudaryti ir išnagrinėti Majoranos spinoriai, turintys realų, o ne labiau įprastą kompleksinį pavidalą. Parodyta, kad tokie spinoriai savyje turi primityvų idempotentą, kuris iš esmės keičia spinoriaus savybes. Gauti spinoriai pritaikyti reliatyvistinės Dirako-Majoranos lygties spektrui – dalelės energijos priklausomybei nuo jos impulso – apskaičiuoti taikant geometrinę $Cl_{1,3}$ algebrą. Išspręstas bėgančios elektroninės bangos uždavinys. Iš sprendinio analizės prieita prie išvados, kad Majoranos tipo laisvosios dalelės neegzistuoja, nes gauto spektro savybės nesiderina su šiuolaikinės fizikos įvaizdžiu, kadangi masę turinčios dalelės greitis negali viršyti arba būti lygus šviesos greičiui.